The Radio Sky (PHY-236.3)
Question Bank: Lectures 1–12

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Questions in the final examination will be similar or identical to the questions in this question bank and in the question bank based on the remaining lectures.

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Constants and conversion factors

Values are rounded to four significant figures. Each problem states whether SI or cgs units are to be used; the coefficients in the last block are given in the unit system shown.

Quantity	Symbol	SI	cgs
Fundamental constants			
Speed of light	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$	$2.998 \times 10^{10}\,\mathrm{cm\,s^{-1}}$
Planck constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$	$6.626 \times 10^{-27}\,\mathrm{erg\,s}$
Boltzmann constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$	$1.381 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$
Elementary charge	e	$1.602 \times 10^{-19}\,\mathrm{C}$	$4.803 \times 10^{-10}\,\mathrm{esu}$
Electron mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg}$	$9.109 \times 10^{-28}\,\mathrm{g}$
Proton mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg}$	$1.673 \times 10^{-24}\,\mathrm{g}$
Hydrogen atom mass	m_H	$1.67 \times 10^{-27}\,\mathrm{kg}$	$1.67 \times 10^{-24}\,\mathrm{g}$
Atomic mass unit	u	$1.6605 \times 10^{-27}\,\mathrm{kg}$	$1.6605 \times 10^{-24}\,\mathrm{g}$
Electron rest energy	$m_\mathrm{e}c^2$	$8.187 \times 10^{-14}\,\mathrm{J}$ (511 keV)	$8.187 \times 10^{-7}\,\mathrm{erg}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$	$6.674 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$
Thomson cross-section	σ_T	$6.652 \times 10^{-29}\,\mathrm{m^2}$	$6.652 \times 10^{-25}\,\mathrm{cm^2}$
Stefan–Boltzmann constant	σ_SB	$5.670 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$	$5.670 \times 10^{-5}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,K^{-4}}$
Radiation constant	$a = 4\sigma_\mathrm{SB}/c$	$7.566 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$	$7.566 \times 10^{-15}\,\mathrm{erg\,cm^{-3}\,K^{-4}}$
Vacuum permittivity	ε_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$	–
Vacuum permeability	μ_0	$1.257 \times 10^{-6}\,\mathrm{H\,m^{-1}}$	–
CMB temperature today	T_CMB	$2.725\,\mathrm{K}$	$2.725\,\mathrm{K}$
Astronomical units and conversions			
Jansky	Jy	$10^{-26}\,\mathrm{W\,m^{-2}\,Hz^{-1}}$	$10^{-23}\,\mathrm{erg\,s^{-1}\,cm^{-2}\,Hz^{-1}}$
Parsec	pc	$3.086 \times 10^{16}\,\mathrm{m}$	$3.086 \times 10^{18}\,\mathrm{cm}$
Megaparsec	Mpc	$3.086 \times 10^{22}\,\mathrm{m}$	$3.086 \times 10^{24}\,\mathrm{cm}$
Astronomical unit	AU	$1.496 \times 10^{11}\,\mathrm{m}$	$1.496 \times 10^{13}\,\mathrm{cm}$
Light year	ly	$9.461 \times 10^{15}\,\mathrm{m}$	$9.461 \times 10^{17}\,\mathrm{cm}$
Solar mass	$M_\odot$	$1.989 \times 10^{30}\,\mathrm{kg}$	$1.989 \times 10^{33}\,\mathrm{g}$
Solar luminosity	$L_\odot$	$3.828 \times 10^{26}\,\mathrm{W}$	$3.828 \times 10^{33}\,\mathrm{erg\,s^{-1}}$
Jupiter radius	R_J	$7.1 \times 10^{7}\,\mathrm{m}$	$7.1 \times 10^{9}\,\mathrm{cm}$
Electron volt	eV	$1.602 \times 10^{-19}\,\mathrm{J}$	$1.602 \times 10^{-12}\,\mathrm{erg}$
Year	yr	$3.156 \times 10^{7}\,\mathrm{s}$	$3.156 \times 10^{7}\,\mathrm{s}$
Gauss	G	$10^{-4}\,\mathrm{T}$	$1\,\mathrm{G}$
Arcsecond	$1''$	$4.848 \times 10^{-6}\,\mathrm{rad}$	$4.848 \times 10^{-6}\,\mathrm{rad}$
Milliarcsecond	mas	$4.848 \times 10^{-9}\,\mathrm{rad}$	$4.848 \times 10^{-9}\,\mathrm{rad}$
Radian in arcseconds	$1\,\mathrm{rad}$	$206265''$	$206265''$
Line and plasma coefficients			
H I line frequency	ν_{10}	$1420.406\,\mathrm{MHz}$	$1420.406\,\mathrm{MHz}$
H I Einstein coefficient	A_{10}	$2.869 \times 10^{-15}\,\mathrm{s^{-1}}$	$2.869 \times 10^{-15}\,\mathrm{s^{-1}}$
H I temperature equivalent	$T_* = h\nu_{10}/k_\mathrm{B}$	$0.0682\,\mathrm{K}$	$0.0682\,\mathrm{K}$
Thin H I column per $\mathrm{K\,km\,s^{-1}}$	N_HI	–	$1.82 \times 10^{18}\,\mathrm{cm^{-2}}$
H I mass per $\mathrm{Mpc^2\,Jy\,km\,s^{-1}}$	M_HI	–	$2.35 \times 10^{5}\,M_\odot$ (slides: 2.36×10^{5})
Electron gyrofrequency coefficient	ν_G/B	$28.0\,\mathrm{GHz\,T^{-1}}$	$2.80\,\mathrm{MHz\,G^{-1}}$
Synchrotron critical frequency	$\nu_\mathrm{c}/(\gamma^2 B)$	–	$4.2\,\mathrm{MHz\,G^{-1}}$
Plasma frequency coefficient	$\nu_\mathrm{p}/\sqrt{n_\mathrm{e}}$	–	$8.98\,\mathrm{kHz\,cm^{3/2}}$
Free-free τ coefficient	–	–	3.28×10^{-7} at $T_\mathrm{e} = 10^4\,\mathrm{K}$, $1\,\mathrm{GHz}$, $\mathrm{EM} = 1\,\mathrm{pc\,cm^{-6}}$
Rotation-measure coefficient	–	–	$0.812\,\mathrm{rad\,m^{-2}}$ per $\mathrm{cm^{-3}}\,\mu\mathrm{G\,pc}$
Dispersion delay coefficient	–	–	$4.149\,\mathrm{ms\,GHz^2}$ per $\mathrm{pc\,cm^{-3}}$

Lecture 1: What is the Radio Sky?

Question 1. Jansky's sidereal hiss

Karl Jansky (1932–33) observed a steady radio hiss at $\lambda = 14.6\,\mathrm{m}$ whose daily peak occurred progressively earlier by clock time. The Earth makes one rotation per sidereal day, and in one tropical year of 365.2422 solar days it makes 366.2422 rotations with respect to the stars.

(a) Derive the length of the sidereal day in hours, minutes and seconds.

(b) By how many seconds per solar day does the peak of a fixed celestial source advance against the clock? After how many days has the peak shifted by 12 h?

(c) Explain why this behaviour shows that the source lies outside the Solar System.

Question 2. The ionospheric cutoff

Radio waves cannot propagate through a plasma at frequencies below the plasma frequency. Use cgs units.

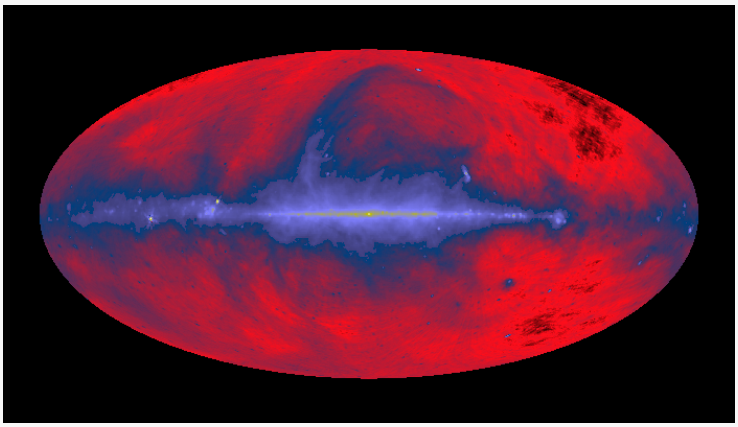
(a) Starting from $\omega_\mathrm{p}^2 = 4\pi n_\mathrm{e} e^2/m_\mathrm{e}$, show that $\nu_\mathrm{p} \approx 8.98\,\mathrm{kHz}\,(n_\mathrm{e}/\mathrm{cm}^{-3})^{1/2}$.

(b) The F-layer of the ionosphere has $n_\mathrm{e} = 10^5$–$10^6\,\mathrm{cm}^{-3}$. Find the corresponding range of ν_p and of the wavelength $\lambda = c/\nu_\mathrm{p}$.

(c) Jansky observed at 20.5 MHz. What is the largest electron density through which this signal could pass?

(d) In one sentence, state what limits the radio window at the high-frequency end.

Question 3. Reading the 408 MHz sky



All-sky radio continuum map at 408 MHz in Galactic coordinates (Galactic Centre at the middle, Galactic plane horizontal).

Answer from the map and the physics covered in the lecture.

(a) What is the dominant diffuse emission mechanism, and what produces the bright horizontal band? Which structures does the map show away from the plane (name two types)?

(b) Name two kinds of bright discrete object that appear against this diffuse background, and say which kind dominates at high Galactic latitude.

(c) Give two reasons why an optical all-sky image looks very different: one about the stellar content, one about dust.

Question 4. Why Reber found a puzzle

Define the Rayleigh–Jeans brightness temperature by $T_\mathrm{b} = c^2 I_\nu/(2k_\mathrm{B}\nu^2)$ and let the source spectrum be $S_\nu \propto \nu^\alpha$ (so $I_\nu \propto \nu^\alpha$ for fixed structure).

(a) Show that $T_\mathrm{b} \propto \nu^{\alpha-2}$.

(b) Grote Reber detected the Milky Way at 160 MHz but found nothing at 900 MHz and 3300 MHz. Suppose the Galactic emission has $\alpha = -0.7$ and $T_{\rm b}(160\,\text{MHz}) = 150\,\text{K}$ (illustrative). Find $T_{\rm b}$ at 900 MHz and 3300 MHz.

(c) An optically thick thermal source has $T_{\rm b} = T_{\rm k}$ independent of frequency, and an optically thin free-free source has $\alpha \approx -0.1$. Compare the fall of $T_{\rm b}$ from 160 MHz to 3300 MHz for these two cases with part (b), and state what feature of the observed emission points to a non-thermal origin.

Question 5. The Ooty Radio Telescope

The ORT is a parabolic cylinder of $530\,\text{m} \times 30\,\text{m}$ operating at $\nu = 326.5\,\text{MHz}$.

(a) Find the wavelength and the diffraction-limited beam widths λ/L along the long axis and across the short axis (in arcmin and degrees respectively).

(b) Find the geometric collecting area, and the number of 45 m GMRT dishes ($A = \pi D^2/4$) that have the same area.

(c) The cylinder axis is parallel to the Earth's rotation axis. Why does the beam then stay on a source as it is tracked, and why is this a design advantage on a hillside of slope equal to the latitude?

Question 6. Diagnosing an unknown source

A source has $S_\nu \propto \nu^{-0.7}$ between 0.3 GHz and 3 GHz, is 20% linearly polarized, and is resolved, with a circular Gaussian size of $30''$ (FWHM). Its flux density at 0.3 GHz is 2 Jy.

(a) Find its flux density at 3 GHz, and use Question 4 to find the factor by which $T_{\rm b}$ changes between the two frequencies.

(b) Using $T_{\rm b} = c^2 S_\nu/(2k_{\rm B}\nu^2\Omega_{\rm s})$ with $\Omega_{\rm s} = \pi\theta^2/(4\ln 2)$, find $T_{\rm b}$ at 0.3 GHz and at 3 GHz.

(c) Is a thermal origin with gas at $T_{\rm k} \sim 10^4\,\text{K}$ allowed? State the most plausible mechanism, and list three further observations needed to classify the source securely.

Lecture 2: GMRT and SKA, Radio Observables and Brightness Temperature

Question 7. GMRT resolution and collecting area

The GMRT has 30 antennas of diameter $D = 45\,\text{m}$, with a maximum baseline $B \approx 25\,\text{km}$. Take the resolution of an interferometer as $\theta \sim \lambda/B$ and the primary beam of a single dish as $\theta \sim \lambda/D$.

(a) Find the array resolution at 150 MHz and at 1.4 GHz (in arcsec).

(b) Find the single-dish primary beam at 1.4 GHz (in arcmin) and the ratio of the two angular scales.

(c) Find the total geometric collecting area of the array, in m^2. Why can the array not image structure larger than about $\lambda/45\,\text{m}$ at 1.4 GHz reliably?

Question 8. From Jy beam^{-1} to brightness temperature

A 1.4 GHz image of a diffuse region has a Gaussian restoring beam with half-power widths $\theta_{\rm maj} = 6''$ and $\theta_{\rm min} = 4''$. A pixel reads $2\,\text{mJy beam}^{-1}$. Use SI units.

(a) Derive the beam solid angle $\Omega_{\rm b} = \int \exp[-4\ln 2\,(\theta/\theta_{\rm HPBW})^2]\,\mathrm{d}\Omega$ for a circular Gaussian of FWHM $\theta_{\rm HPBW}$ and hence the factor $\pi/(4\ln 2)$. Evaluate $\Omega_{\rm b}$ for this beam in sr.

(b) Convert the pixel value to I_ν in Jy sr^{-1} and in $\text{W m}^{-2}\,\text{Hz}^{-1}\,\text{sr}^{-1}$.

(c) Find the brightness temperature of that pixel.

Question 9. The brightness-temperature formula

(a) For a source of flux density S_ν and Gaussian size $\theta_1 \times \theta_2$ (FWHM), starting from $T_\mathrm{b} = c^2 I_\nu/(2k_\mathrm{B}\nu^2)$ and $I_\nu = S_\nu/\Omega_\mathrm{s}$, show that $T_\mathrm{b} \approx 1.22 \times 10^3\,\mathrm{K}\,(S_\nu/\mathrm{mJy})(\nu/\mathrm{GHz})^{-2}(\theta_1\theta_2/\mathrm{arcsec}^2)^{-1}$.

(b) Hence show that in terms of wavelength $T_\mathrm{b} \approx 1.36\,\mathrm{K}\,(\lambda/\mathrm{cm})^2(S_\nu/\mathrm{mJy})(\theta_1\theta_2/\mathrm{arcsec}^2)^{-1}$.

(c) A VLBI observation at 15 GHz of a compact core with $S_\nu = 250\,\mathrm{mJy}$ gives an upper limit on its size of 0.2×0.1 mas. What does this give for T_b? Is it an upper or lower limit, and why?

Question 10. Surface-brightness dimming

(a) Using the fact that I_ν/ν^3 is invariant, show that the bolometric surface brightness of a source at redshift z is $I_\mathrm{obs} = I_\mathrm{em}/(1+z)^4$, and that at a fixed observed frequency $I_\nu^\mathrm{obs}(\nu) = I_\nu^\mathrm{em}(\nu)(1+z)^{\alpha-3}$ for $I_\nu \propto \nu^\alpha$.

(b) Find the dimming factors at $z = 3$ for the bolometric case and for $\alpha = -0.7$ at fixed ν.

(c) A diffuse halo with $\alpha = -0.7$ has $T_\mathrm{b} = 500\,\mathrm{K}$ at 1.4 GHz when seen locally. Find its T_b at 1.4 GHz if the same halo were at $z = 3$.

(d) Why do high-redshift diffuse sources drop out of surveys before compact ones of the same luminosity?

Question 11. Flux density to luminosity

A radio galaxy at $D = 200\,\mathrm{Mpc}$ ($z \ll 1$) has $S_{1.4\,\mathrm{GHz}} = 0.5\,\mathrm{Jy}$ and $S_\nu \propto \nu^{-0.7}$. Use SI units.

(a) Find L_ν at 1.4 GHz in $\mathrm{W\,Hz^{-1}}$ and in $\mathrm{erg\,s^{-1}\,Hz^{-1}}$. Compare with a normal spiral such as the Milky Way ($L_\nu \sim 10^{21}\,\mathrm{W\,Hz^{-1}}$).

(b) Integrate $L_\nu \propto \nu^{-0.7}$ between 0.1 GHz and 10 GHz to get the radio luminosity in this band, in W and in $L_\odot$.

(c) The source is moved to $D = 400\,\mathrm{Mpc}$ with the same L_ν. By what factor does S_ν change? If the radio emission were instead beamed into a solid angle $\Omega_\mathrm{j} = 0.1\,\mathrm{sr}$, by what factor is the true luminosity smaller than the isotropic-equivalent value?

Question 12. Classifying compact sources by T_b

Using the formula derived in Question 9, compute T_b for the three sources below, and say for each which broad class of emission is allowed. Use the thresholds: thermal $T_\mathrm{b} \le T_\mathrm{k} \sim 10^4\,\mathrm{K}$; incoherent synchrotron $T_\mathrm{b} \lesssim 10^{12}\,\mathrm{K}$; above that, coherent emission or relativistic beaming.

Source	S_ν	ν	size (FWHM, or upper limit)
A	300 mJy	1.4 GHz	$60'' \times 60''$
B	250 mJy	15 GHz	0.2 mas $\times$ 0.1 mas (limit)
C	5 Jy	0.4 GHz	1 mas $\times$ 1 mas (limit)

Question 13. What does T_b mean?

Answer in one or two sentences each.

(a) Why is the brightness temperature called "a unit of intensity", and under what condition does it equal a physical temperature?

(b) Why does an unresolved source give only a lower limit on T_b?

(c) Why is the specific intensity of a resolved source, unlike its flux density, independent of distance (for $z \ll 1$)?

(d) Name the physical effect that limits T_b for incoherent synchrotron sources to $\sim 10^{12}\,\mathrm{K}$.

Lecture 3: Source Classification and Radiative Transfer

Question 14. Size from variability

(a) A source varies coherently on a time-scale Δt. Taking $R \lesssim c\Delta t$ and $\Omega_{\rm s} \approx \pi R^2/D^2$, derive $T_{\rm b} \gtrsim S_\nu D^2/(2\pi k_{\rm B}\nu^2\Delta t^2)$ from the Rayleigh–Jeans definition of $T_{\rm b}$. State which unit system you use.

(b) An intraday-variable quasar has $S_\nu = 0.5\,\mathrm{Jy}$ at $\nu = 5\,\mathrm{GHz}$, varies by order unity in $\Delta t = 1\,\mathrm{day}$, and is at $D = 500\,\mathrm{Mpc}$. Evaluate its $T_{\rm b}$ and the size R (in AU).

(c) By what factor would R have to exceed $c\Delta t$ (as a Doppler factor δ relaxes it via $R \lesssim c\delta\Delta t$) for $T_{\rm b}$ to fall to $10^{12}\,\mathrm{K}$? Name one non-intrinsic cause of fast flux variation that could remove the need for this.

Question 15. Coherent emission from a giant pulse

A giant pulse from a pulsar at $D = 2\,\mathrm{kpc}$ has $S_\nu = 2\,\mathrm{kJy}$ at $1.4\,\mathrm{GHz}$ and contains structure lasting $\Delta t = 2\,\mathrm{ns}$.

(a) Using the expression derived in Question 14, find the lower limit on $T_{\rm b}$.

(b) Compare this $T_{\rm b}$ with the incoherent limit $10^{12}\,\mathrm{K}$.

(c) If the emission were incoherent and thermal, the particles would need $k_{\rm B}T \gtrsim k_{\rm B}T_{\rm b}$. Express $k_{\rm B}T_{\rm b}$ in eV and in units of $m_{\rm e}c^2$, and state what this implies for the emission mechanism.

Question 16. Spectral index and spectral shape

A source has flux densities 2.40 Jy at 150 MHz, 0.52 Jy at 1.4 GHz and 0.20 Jy at 5 GHz. Use the convention $S_\nu \propto \nu^{\alpha}$.

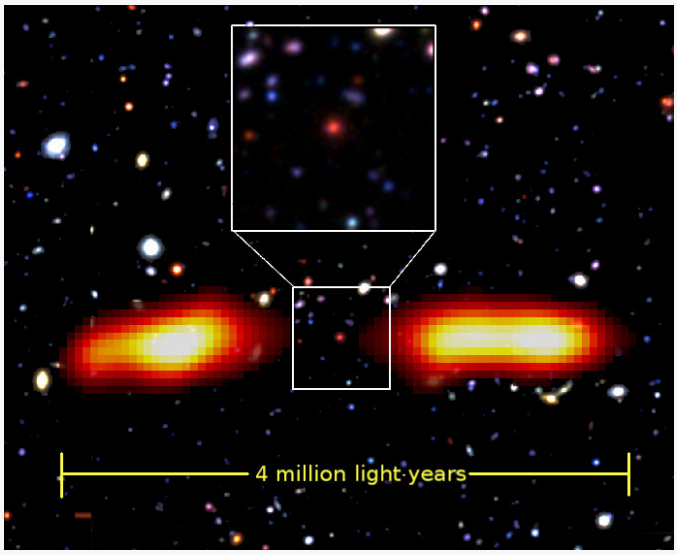
(a) Find α between 150 MHz and 1.4 GHz, and between 1.4 GHz and 5 GHz.

(b) Is the spectrum of a type that could be optically thin thermal free-free? Is there evidence of self-absorption? Which class does it point to?

(c) Using the first spectral index, predict S_ν at 610 MHz.

(d) Using Question 4, by what factor would $T_{\rm b}$ of a source of fixed size change between 150 MHz and 1.4 GHz?

Question 17. Brightness temperature of a giant radio galaxy



GMRT image of a giant radio galaxy: two lobes around a faint host galaxy (optical inset). The scale bar represents 4 million light years.

Assume that the scale bar subtends $8.0'$ on the sky, and that each lobe, measured on the figure,

is an ellipse of FWHM 0.45 of the scale-bar length along the source axis and 0.17 of the scale-bar length across it. Each lobe has $S_\nu = 200\,\mathrm{mJy}$ at 325 MHz. (Illustrative values.)

(a) Find the angular size of a lobe in arcsec, and its physical size in Mpc ($1\,\mathrm{ly} = 9.461\times10^{15}\,\mathrm{m}$).
(b) Use Question 9 to find T_b of a lobe at 325 MHz.
(c) What does T_b imply about the nature of the lobes, and what additional measurement would confirm it?

Question 18. Transfer through a uniform slab

(a) Starting from $\mathrm{d}I_\nu/\mathrm{d}\tau_\nu = -I_\nu + B_\nu(T_\mathrm{ex})$, derive $I_\nu = I_{\nu,0}\mathrm{e}^{-\tau_\nu} + B_\nu(T_\mathrm{ex})(1-\mathrm{e}^{-\tau_\nu})$ for constant T_ex, and its Rayleigh–Jeans form $T_\mathrm{b} = T_\mathrm{bg}\mathrm{e}^{-\tau_\nu} + T_\mathrm{ex}(1-\mathrm{e}^{-\tau_\nu})$.
(b) An H II-like slab with $T_\mathrm{ex} = T_\mathrm{e} = 10^4\,\mathrm{K}$ is seen against $T_\mathrm{bg} = 2.73\,\mathrm{K}$. Find T_b for $\tau_\nu = 0.01,\ 0.1,\ 1,\ 3$.
(c) A cold cloud with $T_\mathrm{ex} = 100\,\mathrm{K}$ and $\tau_\nu = 0.5$ lies in front of a background source with $T_\mathrm{bg} = 3000\,\mathrm{K}$. Find $\Delta T_\mathrm{b} = T_\mathrm{b} - T_\mathrm{bg}$ and the observed T_b. Is this emission or absorption, and what sets the sign?

Question 19. When is Rayleigh–Jeans safe?

Outside the Rayleigh–Jeans regime, a blackbody at temperature T has $T_\mathrm{b} = T\,x/(\mathrm{e}^x-1)$ with $x = h\nu/k_\mathrm{B}T$, because T_b is always defined with the RJ factor.

(a) Show that for a blackbody $x = 0.048\,(\nu/\mathrm{GHz})/(T/\mathrm{K})$ and verify that $x/(\mathrm{e}^x-1) = 0.55$ for CO 2–1 ($\nu = 230\,\mathrm{GHz}$) in gas at $T = 10\,\mathrm{K}$.
(b) Optically thick gas observed at 230 GHz shows $T_\mathrm{b} = 5\,\mathrm{K}$. Invert the relation to find its true temperature.
(c) Find T_b of the 2.725 K CMB at 100 GHz, and the factor by which it is below 2.725 K.

Question 20. Classification of radio emission

(a) List the four axes along which radio emission is classified, naming the two ends of each.
(b) Place each of: an H II region, a pulsar, an astrophysical maser, and a radio-galaxy lobe on these axes (one line each).
(c) Give two examples where a single diagnostic (spectral index, polarization or T_b) would mislead if used alone.
(d) State Kirchhoff's law and the thermal ceiling it implies on T_b; name one situation where j_ν/κ_ν is not $B_\nu(T_\mathrm{k})$.

Lecture 4: Free-Free Emission and Synchrotron Radiation

Question 21. Free-free versus synchrotron: why one is thermal

Free-free and synchrotron radiation are both produced by accelerated electrons, yet only free-free emission from an ionized nebula obeys $S_\nu = B_\nu(T_\mathrm{e})$.

(a) State, for each process, the force that accelerates the electron and the energy distribution of the radiating electrons.
(b) Give three reasons why radio free-free emission comes mainly from distant, weak electron–ion encounters and why electron–electron encounters contribute no dipole radiation.
(c) Explain in two or three sentences why Kirchhoff's law with a single temperature cannot be applied to synchrotron emission.

Question 22. Gyration, beaming and the critical frequency

An electron with Lorentz factor $\gamma = 3000$ gyrates in a uniform magnetic field $B = 20\,\mu\mathrm{G}$ with pitch angle $\vartheta = 60^\circ$ (cgs units; $e = 4.803 \times 10^{-10}\,\mathrm{esu}$).

(a) Compute the non-relativistic gyrofrequency $\nu_\mathrm{G} = eB/2\pi m_\mathrm{e} c$ and the relativistic orbital period $T = \gamma/\nu_\mathrm{G}$.

(b) The emission is beamed into a cone of half-angle $\sim 1/\gamma$. Argue that the observer sees a pulse of duration $\Delta t \sim T/\gamma^3$, and hence that the radiation extends to $\nu \sim \gamma^2 \nu_\mathrm{G}$.

(c) Evaluate Δt and $1/\Delta t$ numerically, and compare with the critical frequency $\nu_\mathrm{c} = \frac{3}{2}\gamma^2 \nu_\mathrm{G} \sin\vartheta$.

Question 23. Synchrotron power and cooling time

Consider the electron of Question 22 ($\gamma = 3000$, $B = 20\,\mu\mathrm{G}$, $\vartheta = 60^\circ$). Use SI units: $1\,\mathrm{G} = 10^{-4}\,\mathrm{T}$, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$, $\sigma_\mathrm{T} = 6.652 \times 10^{-29}\,\mathrm{m^2}$.

(a) Find the magnetic energy density $U_B = B^2/2\mu_0$ and the radiated power $P = 2\sigma_\mathrm{T} c\,\gamma^2\beta^2 U_B \sin^2\vartheta$.

(b) Estimate the radiative lifetime $t = \gamma m_\mathrm{e} c^2/P$ in years.

(c) Average $\sin^2\vartheta$ over isotropically distributed pitch angles to obtain the $\frac{4}{3}$ in $\langle P\rangle = \frac{4}{3}\sigma_\mathrm{T} c \gamma^2\beta^2 U_B$, and find the lifetime for such an isotropic population.

Question 24. Spectrum of a power-law electron population

Electrons have $N(E)\,\mathrm{d}E = K E^{-p}\,\mathrm{d}E$ in a uniform field B. Treat each electron as radiating only at its own critical frequency, $\nu \propto E^2 B$, with power $P \propto E^2 B^2$ (fixed pitch angle).

(a) Starting from $j_\nu\,\mathrm{d}\nu \propto P(E)N(E)\,\mathrm{d}E$, show that $j_\nu \propto K\,B^{(p+1)/2}\,\nu^{-(p-1)/2}$.

(b) A source has $S_\nu \propto \nu^{-0.7}$ in the thin regime. Find p and the maximum fractional linear polarization $\Pi_\mathrm{max} = (p+1)/(p+7/3)$.

(c) The field is revised upwards by a factor 2 with the observed j_ν unchanged. By what factor must K change?

Question 25. Free-free optical depth and turnover frequency

Use $\tau_\nu \approx 3.28 \times 10^{-7}\,(T_\mathrm{e}/10^4\,\mathrm{K})^{-1.35}(\nu/\mathrm{GHz})^{-2.1}\,(\mathrm{EM}/\mathrm{pc\,cm^{-6}})$.

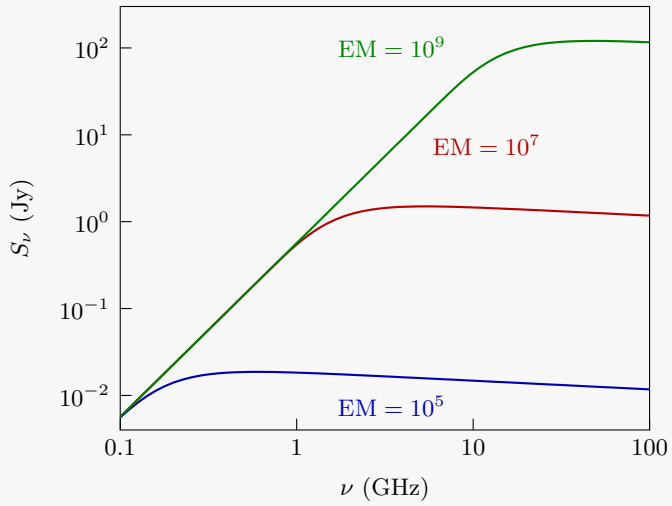
(a) Setting $\tau_\nu = 1$, derive ν_t in terms of T_e and EM, and give the power-law dependences on T_e and EM.

(b) Evaluate ν_t for $T_\mathrm{e} = 8000\,\mathrm{K}$ and $\mathrm{EM} = 5 \times 10^7\,\mathrm{pc\,cm^{-6}}$ (use the exact power $1/2.1$).

(c) For the same region, find τ_ν and the brightness temperature $T_\mathrm{b} = T_\mathrm{e}(1 - \mathrm{e}^{-\tau_\nu})$ at 1.4 GHz and at 15 GHz.

Question 26. Where does the free-free flux peak?

The figure shows $S_\nu = (2k_\mathrm{B}T_\mathrm{e}\nu^2/c^2)\,\Omega_\mathrm{s}(1-\mathrm{e}^{-\tau_\nu})$ for a uniform $10''$-diameter disc with $T_\mathrm{e} = 10^4\,\mathrm{K}$ and three emission measures, with $\tau_\nu \propto \nu^{-2.1}$.



Free-free spectrum of a uniform disc of diameter $10''$, $T_e = 10^4$ K; EM in pc cm^{-6}.

(a) Show that S_ν peaks where $\tau/(e^\tau - 1) = 2/2.1$, and that this gives $\tau \approx 0.1$. Deduce $\nu_{\rm peak}/\nu_{\rm t}$.
(b) Using $\nu_{\rm t}$ from Question 25(a) for EM $= 10^7$, predict $\nu_{\rm peak}$ and compare with the figure.
(c) All three curves coincide at low frequency. Why, and what is the flux density at 0.1 GHz? Check with the Rayleigh–Jeans law ($1\,\mathrm{Jy} = 10^{-23}\,\mathrm{erg\,s^{-1}cm^{-2}Hz^{-1}}$, $\Omega_{\rm s} = \pi(5'')^2$).

Lecture 5: Spectral Signatures and Free-Free Diagnostics

Question 27. Emission or absorption? Brightness temperature against optical depth

A uniform cloud of excitation temperature $T_{\rm ex} = 100$ K has optical depth τ and lies in front of a background of brightness temperature $T_{\rm bg}$.
(a) Write the observed $T_{\rm b}$ and the contrast $\Delta T_{\rm b} = T_{\rm b} - T_{\rm bg}$ in terms of $T_{\rm ex}$, $T_{\rm bg}$ and τ.
(b) Evaluate $T_{\rm b}$ and $\Delta T_{\rm b}$ for $T_{\rm bg} = 2.7$ K and $T_{\rm bg} = 300$ K, each for $\tau = 0.5$ and $\tau = 3$. State whether the cloud is seen in emission or absorption in each case.
(c) Why does the observation at $\tau = 3$ carry little information on the column density of the cloud?

Question 28. Telling turnovers apart

A radio spectrum rises at low frequency and turns over. List five physical origins of such a turnover. For each give, in one phrase, the low-frequency spectral index ($S_\nu \propto \nu^\alpha$) or shape, and one additional observable that distinguishes it from the others. Why is the angular size often the decisive measurement?

Question 29. Temperature and emission measure of a resolved H II region

A uniform, isothermal H II region is resolved, so $T_{\rm b}$ can be measured: $T_{\rm b} = 8000$ K at 0.2 GHz and $T_{\rm b} = 60$ K at 15 GHz. Use $\tau_\nu = 3.28 \times 10^{-7} T_4^{-1.35} \nu_{\rm GHz}^{-2.1}\,\mathrm{EM}$ with $T_4 = T_{\rm e}/10^4$ K.
(a) What do you conclude for $T_{\rm e}$ from the 0.2 GHz measurement, and why?
(b) Find τ at 15 GHz and invert the formula for EM (in pc cm^{-6}).
(c) Predict the turnover frequency $\nu_{\rm t}$ (using Question 25(a)) and the brightness temperature there. Why would the thin approximation $T_{\rm b} \approx T_{\rm e}\tau$ be a poor way to analyse a 3 GHz measurement of this source?

Question 30. An unresolved compact H II region

An unresolved thermal source at $D = 5\,\mathrm{kpc}$ has $S_\nu = 0.036\,\mathrm{Jy}$ at $1.4\,\mathrm{GHz}$ on a thick rising spectrum with $\alpha = +2$, and a turnover ($\tau_\nu = 1$) at $\nu_\mathrm{t} = 10\,\mathrm{GHz}$. Assume a uniform isothermal disc with $T_\mathrm{e} = 8000\,\mathrm{K}$ from a recombination line (cgs: $k_\mathrm{B} = 1.381 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$, $1\,\mathrm{Jy} = 10^{-23}\,\mathrm{erg\,s^{-1}cm^{-2}Hz^{-1}}$).

(a) Find the solid angle Ω_s and angular diameter of the disc from the thick-side flux density.

(b) Invert the turnover condition, as in Question 29, to get EM.

(c) Taking the path length equal to the physical diameter, find $n_\mathrm{rms} = \sqrt{\mathrm{EM}/L}$ and classify the source. Which assumption is needed for (a) but not for (b)?

Question 31. H I absorption against a continuum source

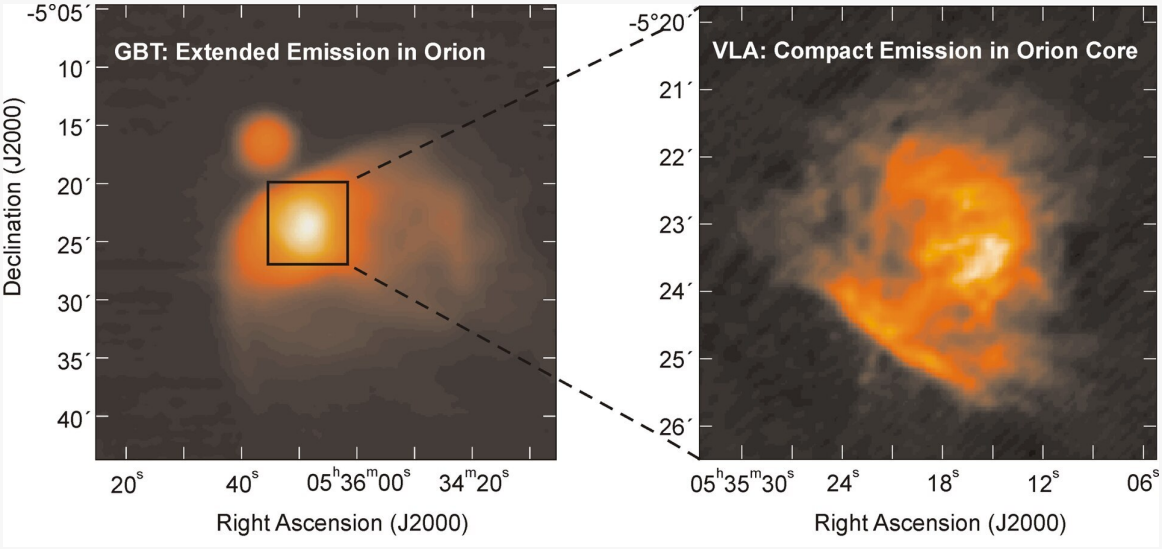
A cold H I cloud lies in front of a compact continuum source with brightness temperature $T_\mathrm{c} = 1000\,\mathrm{K}$ (measured from line-free channels). At line centre the on-source spectrum minus the off-source spectrum is $T_\mathrm{on} - T_\mathrm{off} = 370\,\mathrm{K}$, and the off-source spectrum (cloud emission only) is $T_\mathrm{off} = 40\,\mathrm{K}$.

(a) Using $T_\mathrm{on} - T_\mathrm{off} = T_\mathrm{c}\,\mathrm{e}^{-\tau}$, find τ at line centre.

(b) Find the spin temperature T_s from $T_\mathrm{off} = T_\mathrm{s}(1 - \mathrm{e}^{-\tau})$.

(c) The fractional depth $1 - \mathrm{e}^{-\tau}$ is measured to ± 0.01. Compare the resulting uncertainty in τ for $\tau = 0.5$ and for $\tau = 4$. What goes wrong at large τ?

Question 32. Reading the Orion Nebula at 3.6 cm



GBT (left) and VLA (right) 3.6 cm continuum of the Orion Nebula. NRAO/AUI; D. Shepherd, R. Maddalena, J. McMullin.

Adopt a distance of 414 pc, $T_\mathrm{e} = 8000\,\mathrm{K}$, and an emission measure $\mathrm{EM} = 4 \times 10^6\,\mathrm{pc\,cm^{-6}}$ for the bright core.

(a) In the VLA panel the bright emission spans declination from about $-5°22'$ to $-5°25.5'$. Convert to a physical size L in pc.

(b) Show that the core is optically thin at 3.6 cm ($\approx 8.3\,\mathrm{GHz}$) and find n_rms. If the gas fills only a fraction $f = 0.2$ of the path, find the local n_e, and say by what factor a smooth ($f = 1$) model overestimates the ionized mass.

(c) The GBT panel shows smooth extended emission and the VLA panel only the compact structure. Why do both panels appear and what does each instrument miss?

Question 33. Spectrum of an ionized stellar wind

In a spherical wind of constant speed the electron density is $n_{\rm e} \propto r^{-2}$. Take $T_{\rm e}$ constant and $\tau_\nu \propto \nu^{-2.1}\mathrm{EM}$.

(a) Show that for a line of sight at impact parameter b, $\mathrm{EM}(b) \propto b^{-3}$, and hence that the radius r_ν at which $\tau_\nu = 1$ scales as $\nu^{-0.7}$.

(b) Treating the source as a thick disc of radius r_ν with $T_{\rm b} = T_{\rm e}$, show that $S_\nu \propto \nu^{0.6}$.

(c) Compare the flux density and angular size at 15 GHz with those at 5 GHz. Why is this spectrum shallower than ν^2?

Lecture 6: Synchrotron Polarization, Faraday Rotation and Dispersion

Question 34. What a polarization map does and does not tell you

(a) Explain why only the field component $B_\perp = B\sin\vartheta$ perpendicular to the line of sight matters for synchrotron emission, and why $B_\parallel$ is invisible in the emission.

(b) State the orientation of the electric vector relative to the sky-projected field, and what a map of polarization vectors therefore shows. Name two properties of the field that the map cannot give.

(c) Young supernova-remnant shells tend to show radial and older ones tangential projected fields. Give one proposed physical origin for each, and one caution about reading such a map.

Question 35. Which electrons produce the radio emission?

Invert the critical-frequency relation of Question 22, $\nu \approx \nu_{\rm c} = 4.2\,\mathrm{MHz}\,\gamma^2 (B/\mathrm{G})\sin\vartheta$, for a source with $B = 5\,\mu\mathrm{G}$.

(a) Find γ and the electron energy $E = \gamma m_{\rm e} c^2$ (in GeV) responsible for emission at 150 MHz and at 1.4 GHz, with $\vartheta = 90^\circ$.

(b) Repeat for 1.4 GHz with $\vartheta = 30^\circ$.

(c) By what factor does γ increase if one uses the spectrum peak $\nu = 0.29\,\nu_{\rm c}$ instead of $\nu = \nu_{\rm c}$? What does this say about the radio sky and cosmic-ray electrons near Earth?

Question 36. Faraday rotation through the interstellar medium

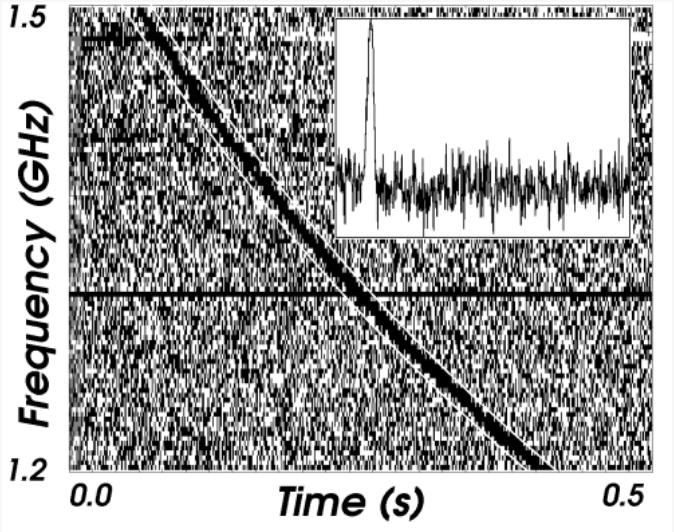
A polarized source lies behind a path $L = 1\,\mathrm{kpc}$ through gas with uniform $n_{\rm e} = 0.03\,\mathrm{cm}^{-3}$ and $B_\parallel = 2\,\mu\mathrm{G}$. Use $\mathrm{RM} = 0.812 \int (n_{\rm e}/\mathrm{cm}^{-3})(B_\parallel/\mu\mathrm{G})\,(\mathrm{d}l/\mathrm{pc})\,\mathrm{rad\,m^{-2}}$ and $\chi = \chi_0 + \mathrm{RM}\,\lambda^2$.

(a) Compute RM.

(b) Find the rotation $\mathrm{RM}\,\lambda^2$, in degrees, at 6 cm and at 21 cm, and in radians at 2 m.

(c) At 21 cm the observed angle is $\chi = 40^\circ$. What is χ_0 (modulo 180°)? Why can RM not be found from a single wavelength, and why are several wavelengths needed?

Question 37. Dispersion measure of the Lorimer burst



Frequency–time plot of a single radio pulse recorded with Parkes between 1.2 and 1.5 GHz (inset: the dedispersed pulse). The Planetary Society; Lorimer et al. 2007.

Use $t_{\rm DM} = 4.149\,\mathrm{ms}\,(\mathrm{DM}/\mathrm{pc\,cm^{-3}})\,(\nu/\mathrm{GHz})^{-2}$.

(a) Read from the figure the delay of the pulse between 1.5 and 1.2 GHz, and use it to estimate DM.

(b) If the intervening gas had a mean $n_{\rm e} = 0.03\,\mathrm{cm^{-3}}$, how long would the path be? What does the answer imply about the source?

(c) With 3 MHz channels at 1.4 GHz, what is the dispersion smearing within one channel? Why is this a problem for detecting the burst?

Question 38. EM, DM and RM of one slab

A uniform ionized slab has $n_{\rm e} = 0.1\,\mathrm{cm^{-3}}$, thickness $L = 100\,\mathrm{pc}$, $B_\parallel = 3\,\mu\mathrm{G}$ and $T_{\rm e} = 8000\,\mathrm{K}$.

(a) Compute EM, DM and RM.

(b) Using the free-free optical depth of Question 25, find τ_ν at 150 MHz.

(c) Find the pulse delay between 600 and 300 MHz, and the Faraday rotation $\mathrm{RM}\,\lambda^2$ at 150 MHz (in rad).

(d) Check that $1.232\,\mu\mathrm{G\,RM/DM}$ returns $B_\parallel$.

Question 39. Fractional polarization of a real source

A synchrotron source has a thin spectrum $S_\nu \propto \nu^{-0.8}$.

(a) Using the results of Question 24, find p and the maximum polarization $\Pi_{\rm max}$ for a uniform field.

(b) The field has a uniform sky-plane part $B_{0\perp}$ and an isotropic random part with $\langle b_\perp^2 \rangle = B_{0\perp}^2$. Using $\Pi \approx \Pi_{\rm max} B_{0\perp}^2/(B_{0\perp}^2 + \langle b_\perp^2 \rangle)$, find Π.

(c) The emission also passes through a uniform slab with $\mathrm{RM} = 40\,\mathrm{rad\,m^{-2}}$ in which emission and rotation are mixed, so that $\Pi = \Pi_0\,|\sin(\mathrm{RM}\lambda^2)/(\mathrm{RM}\lambda^2)|$. Find the net Π at 21 cm and 50 cm (assume the two effects multiply), and the first wavelength at which $\Pi \to 0$.

Question 40. Ways to depolarize synchrotron emission

Observed polarization fractions are far below $\Pi_{\rm max} \approx 70\%$. List the four depolarization mechanisms discussed in the lectures, with one sentence each on the physics, state which of them depends on wavelength, and say how each can be reduced observationally. Which of these was the cause of the reduction in Question 39(b) and (c)?

Lecture 7: Radio Galaxies, Magnetic Fields and Synchrotron Ageing

Question 41. Debye screening in three plasmas

Free charges screen a static electric field beyond the Debye length $\lambda_{\rm D} = \sqrt{\varepsilon_0 k_{\rm B} T_{\rm e}/(n_{\rm e} e^2)}$ (SI units). Take $\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$, and the plasma frequency $\nu_{\rm p} = 8.98\,\mathrm{kHz}\,(n_{\rm e}/\mathrm{cm^{-3}})^{1/2}$.

(a) Calculate $\lambda_{\rm D}$ for warm interstellar gas ($T_{\rm e} = 8 \times 10^3\,\mathrm{K}$, $n_{\rm e} = 0.3\,\mathrm{cm^{-3}}$).

(b) Calculate $\lambda_{\rm D}$ for hot cluster gas ($T_{\rm e} = 10^8\,\mathrm{K}$, $n_{\rm e} = 10^{-3}\,\mathrm{cm^{-3}}$), and compare it with a radio lobe of size 100 kpc.

(c) Estimate the time $1/\nu_{\rm p}$ in which the warm interstellar gas cancels a charge imbalance. Hence explain in two sentences why we can set $\mathbf{E} \approx 0$ in the plasma frame in radio sources, and what accelerates the electrons instead.

Question 42. Particle–field degeneracy of the synchrotron flux

For a power-law electron population $N(E)\,\mathrm{d}E = K E^{-p}\,\mathrm{d}E$ a single electron radiates power $P \propto E^2 B^2$, mostly at $\nu \approx \gamma^2 \nu_{\rm G} \propto E^2 B$.

(a) Show that $j_\nu \propto K\, B^{(p+1)/2}\, \nu^{-(p-1)/2}$, and give the spectral index α in terms of p.

(b) An optically thin source with $p = 2.4$ has volume V and distance D fixed. Its flux density is measured. A second model has a field five times stronger. By what factor must K change to give the same S_ν?

(c) Does a map of the polarization angle break this degeneracy? Answer in one sentence.

Question 43. Why electrons, and why magnetic fields

EM, DM and RM trace only electrons, and synchrotron emission requires a magnetic field. Answer concisely.

(a) Using $a = qE/m$ and the scalings DM $\propto q^2/m$, RM $\propto q^3 B/m^2$ and radiated power $\propto a^2$, evaluate the ratio (proton contribution)/(electron contribution) for each of the three, using $m_{\rm p}/m_{\rm e} = 1836$.

(b) Why does a charge moving at constant velocity in empty space not radiate, even though its electric field is Lorentz-contracted into a thin disc?

(c) Why is a relativistic electron in a magnetic field able to radiate for a long time without gaining energy, while a static electric field in a plasma cannot do the same?

Question 44. Radiative lifetime of electrons in a lobe

Electrons with $\gamma \gg 1$ lose energy as $\mathrm{d}\gamma/\mathrm{d}t = -b\gamma^2$, $b = \dfrac{4\sigma_{\rm T}}{3 m_{\rm e} c}(U_B + U_{\rm CMB})$ (SI units), with $U_B = B^2/2\mu_0$ and $U_{\rm CMB} = a T_0^4 (1+z)^4$. Take $a = 7.566 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$, $T_0 = 2.725\,\mathrm{K}$, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$ and $\nu_{\rm G} = eB/2\pi m_{\rm e}$. A lobe at $z = 0$ has $B = 10\,\mu\mathrm{G}$ ($1\,\mathrm{G} = 10^{-4}\,\mathrm{T}$).

(a) Calculate U_B, $U_{\rm CMB}$ and b.

(b) Find the Lorentz factor of the electrons that radiate mostly at $\nu = 1.4\,\mathrm{GHz}$, using $\gamma = (\nu/\nu_{\rm G})^{1/2}$.

(c) Calculate their cooling time $t_{\rm cool} = 1/(b\gamma)$ in Myr.

Question 45. The longest-lived field

Using the cooling time of Question 44, in which $t_{\rm cool} \propto B^{1/2}/(U_B + U_{\rm CMB}) \propto B^{1/2}/(B^2 + B_{\rm IC}^2)$ at fixed observing frequency:

(a) Define $B_{\rm IC}$ by $U_B = U_{\rm CMB}$ and calculate it at $z = 0$ in μG.

(b) Show that for a given ν and z the cooling time is maximum for $B = B_{\rm IC}/\sqrt{3}$, and calculate this B at $z = 0$ and $z = 1$.

(c) Evaluate the maximum cooling time at $\nu = 1.4\,\mathrm{GHz}$ for $z = 0$ and $z = 1$ (rest-frame frequency 1.4 GHz). Comment on what happens at higher redshift.

Question 46. Break frequency, spectral age and a dying radio galaxy

Electrons injected once with γ_0 and cooling as in Question 44.

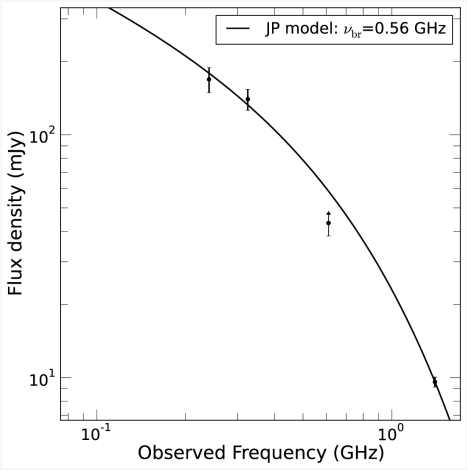
(a) Solve $\mathrm{d}\gamma/\mathrm{d}t = -b\gamma^2$ for $\gamma(t)$, and show that after a time t no electron has $\gamma > \gamma_{\max} = 1/(bt)$ whatever γ_0 was. Hence show that the spectrum steepens above $\nu_\mathrm{b} = \nu_\mathrm{G}/(bt)^2$.

(b) For $B = 5\,\mu\mathrm{G}$ at $z = 0$, calculate ν_b after $t = 10\,\mathrm{Myr}$ and after 30 Myr.

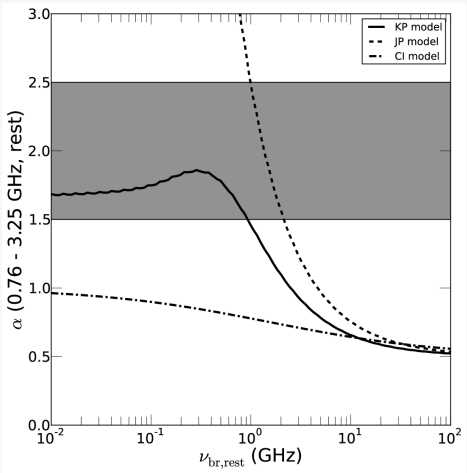
(c) A relic at $z = 1.33$ has a fitted observed break 0.57 GHz and $B = 3.3\,\mu\mathrm{G}$. Calculate the rest-frame break and the spectral age in Myr. Which loss process dominates, and by what factor in energy density?

Use $U_\mathrm{CMB} = 4.17 \times 10^{-14}\,\mathrm{J\,m^{-3}}(1+z)^4$ and the other constants of Question 44.

Question 47. Reading the spectrum and injection models of a relic



Integrated flux density of the relic J021659−044920 ($z = 1.33$) with a JP-model fit.



Spectral steepness $-\alpha$ between rest-frame 0.76 and 3.25 GHz against rest-frame break frequency, for three injection models. Grey band: the range observed across the lobes ($-\alpha = 1.5$–2.5).

(a) From the first figure read S_ν at 0.325 GHz and 1.4 GHz (nearest data points) and calculate the spectral index α ($S_\nu \propto \nu^\alpha$). Convert the quoted break to the rest frame.

(b) From the second figure, which model(s) cannot reach the grey band at any break frequency? Which reaches it only for $\nu_{\mathrm{br,rest}} \lesssim 0.9\,\mathrm{GHz}$? Give the range of $\nu_{\mathrm{br,rest}}$ (and the corresponding observed frequencies) for which JP spans the band.

(c) In a sentence each, explain physically why the CI model never gets steeper than $\alpha_{\rm inj} - 0.5$ and why JP gives an exponential cutoff.

Lecture 8: Synchrotron Self-Absorption and Equipartition

Question 48. The self-absorbed synchrotron spectrum

In a synchrotron source with $N(E) = KE^{-p}$, the emissivity and absorption coefficient are $j_\nu = \frac{1}{4\pi}\int P(\nu,E)KE^{-p}\,{\rm d}E$ and $\kappa_\nu = \frac{c^2}{8\pi\nu^2}(p+2)\int P(\nu,E)KE^{-p-1}\,{\rm d}E$, where $P(\nu,E)$ is sharply peaked at the energy $E_* = \gamma m_{\rm e}c^2$ with $\nu \approx \gamma^2\nu_{\rm G}$.

(a) Show that the source function is $S_\nu = j_\nu/\kappa_\nu = 2k_{\rm B}T_{\rm eff}\nu^2/c^2$ with $k_{\rm B}T_{\rm eff} = \gamma m_{\rm e}c^2/(p+2)$.

(b) Show that for a source of angular size θ the optically thick flux density scales as $S_\nu \propto \theta^2 B^{-1/2}\nu^{5/2}$, and say why the slope does not depend on p.

(c) Calculate $T_{\rm eff}$ at $\nu = 5\,{\rm GHz}$ for $B = 0.16\,{\rm G}$ and $p = 2.4$ (cgs; $\nu_{\rm G} = 2.80\,{\rm MHz}\,(B/{\rm G})$).

Question 49. Field in a compact peaked-spectrum source

A compact source has a spectral peak $S_{\rm p} = 1\,{\rm Jy}$ at $\nu_{\rm p} = 5\,{\rm GHz}$ and a Gaussian FWHM size $\theta = 1\,{\rm mas}$ from VLBI; take $z \approx 0$ and no beaming ($\delta = 1$).

(a) Use $T_{\rm b} = 1.22\times10^3\,{\rm K}\,(S/{\rm mJy})(\nu/{\rm GHz})^{-2}(\theta/{\rm arcsec})^{-2}$ to find $T_{\rm b}$ at the peak. Is it consistent with incoherent synchrotron emission?

(b) Use $B \approx 5\times10^{-5}\,(\theta/{\rm mas})^4(\nu_{\rm p}/{\rm GHz})^5(S_{\rm p}/{\rm Jy})^{-2}\,\delta/(1+z)$ G to find B. By what factor does B change if θ is really twice as large?

(c) Using $b = \frac{4\sigma_{\rm T}}{3m_{\rm e}c}U_B$ ($U_{\rm CMB}$ negligible) and the method of Question 44, find the cooling time of the electrons that radiate at 5 GHz. What do you conclude about the age of the source?

Question 50. Self-absorption or free-free absorption?

A compact source has the flux densities below at low frequencies.

ν (GHz)	0.12	0.15	0.24
S_ν (mJy)	61	299	1250

(a) Calculate the spectral slope between consecutive points. What is the maximum low-frequency slope a homogeneous synchrotron self-absorbed source can have? What do you conclude?

(b) For a foreground screen, $S_\nu \propto \nu^\alpha {\rm e}^{-\tau}$ with $\tau \propto \nu^{-2.1}$. Show that the local slope is $\alpha + 2.1\tau$. For $\alpha = -0.7$, at what τ does the slope exceed $+5/2$?

(c) With $\tau^{\rm ff} = 3.28\times10^{-7}\,(T_{\rm e}/10^4\,{\rm K})^{-1.35}(\nu/{\rm GHz})^{-2.1}({\rm EM/pc\,cm^{-6}})$, find the emission measure that gives $\tau = 1$ at 0.25 GHz for $T_{\rm e} = 10^4\,{\rm K}$.

Question 51. Equipartition and the energy of a lobe

At fixed radio luminosity L the particle energy is $U_{\rm part} = AB^{-3/2}$, and the field energy is $U_B = B^2\phi V/8\pi$ (cgs), where A is independent of B.

(a) Minimize $U_{\rm part} + U_B$ with respect to B and show that $U_B = \frac{3}{4}U_{\rm part}$ and $U_{\rm min} = \frac{7}{24\pi}B_{\rm eq}^2\phi V$.

(b) Given $B_{\rm eq} \propto (L/\phi V)^{2/7}$, how does $B_{\rm eq}$ change if the source volume is 10 times larger at the same L?

(c) A relic lobe system consists of two cylinders of total length 1 Mpc and diameter 180 kpc ($\phi = 1$), with $B = 3.3\,\mu{\rm G}$. Calculate U_B and the particle energy predicted by equipartition. (The X-ray luminosity implies $U_{\rm e} \gtrsim 4.2\times10^{59}$ erg: comment.)

Question 52. Magnetic field from inverse-Compton X-rays

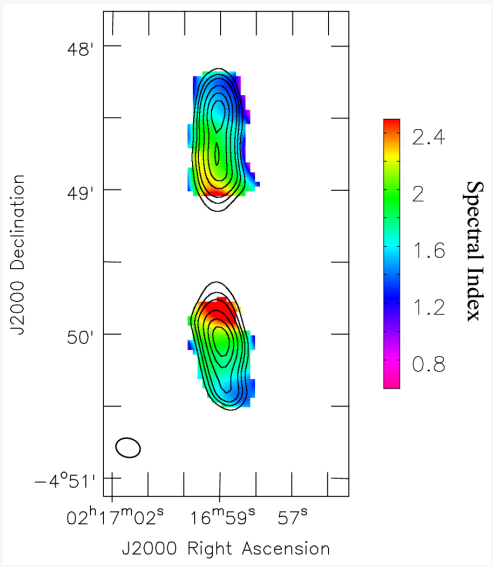
The electrons in a lobe radiate synchrotron photons in B and inverse-Compton photons off the CMB.

(a) Using $P_{\rm IC}/P_{\rm syn} = U_{\rm rad}/U_B$ and $B_{\rm IC} = 3.24(1+z)^2\,\mu$G, show that $L_{\rm IC}/L_{\rm syn} = (B_{\rm IC}/B)^2$.

(b) At $z = 0.5$ the inverse-Compton luminosity of a lobe is 4 times its synchrotron luminosity (same electrons; illustrative). Find B. Why is this estimate independent of equipartition?

(c) For the relic of Question 46 ($z = 1.33$) an electron of Lorentz factor γ scatters a CMB photon of energy $\epsilon = 2.7k_{\rm B}T$ to $\langle E\rangle = \frac{4}{3}\gamma^2\epsilon$. Find γ for an observed X-ray energy of 1 keV, and compare with the γ of the electrons that radiate at an observed 0.325 GHz in $B = 3.3\,\mu$G (cgs; $\nu_{\rm G} = eB/2\pi m_{\rm e}c$).

Question 53. Spectral-index map of a relic



Spectral-index map between 0.325 and 1.4 GHz of the relic J021659−044920 (contours: 325 MHz). The colour bar shows $-\alpha$ ($S_\nu \propto \nu^\alpha$).

(a) Read the approximate values of α at the outer tips of the two lobes and at their inner ends (nearest the host, midway between them). By what factor does the ratio $S_{1.4}/S_{0.325}$ differ between those two places?

(b) Explain physically where the oldest electrons are, and why the 325-MHz lobes extend further than the 1.4-GHz emission.

(c) The age from Question 46 used $B = 3.3\,\mu$G with $B_{\rm IC} = 17.6\,\mu$G. If the true B were twice as large, by what factor would the cooling time at a fixed frequency change? Compare with the naive synchrotron-only scaling and say which is correct here.

Lecture 9: Gyro-Emission, Electron-Cyclotron Masers and Coherent Radio Bursts

Question 54. Gyrofrequency, harmonics and field from a burst

(a) Calculate $\nu_{\rm G} = eB/(2\pi m_{\rm e}c)$ (cgs) in MHz per gauss.

(b) A pulsed burst from an ultracool dwarf is seen at 8.4 GHz. What field is needed if the emission is at the fundamental ($s = 1$) or at the second harmonic ($s = 2$)? Why is this a lower limit on the surface field?

(c) The emitting electrons have kinetic energy 10 keV and move perpendicular to the line of

sight along B ($\beta_\parallel \cos\theta = 0$). Calculate γ and the fractional shift of the $s = 1$ emission from $\nu_{\rm G}$, using $\nu = s\nu_{\rm G}/\gamma(1 - \beta_\parallel \cos\theta)$.

Question 55. Is the brightness temperature enough?

An ultracool dwarf at $d = 10\,\mathrm{pc}$ emits a pulse of $S_\nu = 1\,\mathrm{mJy}$ at $\nu = 8\,\mathrm{GHz}$. Use SI units; $R_{\rm J} = 7.1 \times 10^7\,\mathrm{m}$.

(a) Show that in the Rayleigh–Jeans limit $T_{\rm b} = c^2 S_\nu/(2k_{\rm B}\nu^2\Omega_{\rm s})$ and calculate it for an emitting disc of radius $R_{\rm J}$ ($\Omega_{\rm s} = \pi R^2/d^2$).

(b) Recalculate $T_{\rm b}$ for an emitting radius $0.1R_{\rm J}$.

(c) An incoherent gyrosynchrotron source is limited by $T_{\rm b} \lesssim E/k_{\rm B}$. Evaluate this for $E = 1\,\mathrm{MeV}$ ($1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$), and find the largest emitting radius for which the observed pulse *requires* coherent emission. What other observations would you use to decide?

Question 56. Amplification in an inverted medium

In the Rayleigh–Jeans limit the radiative transfer equation of Lecture 3 gives $T_{\rm b} = T_{\rm b}(0)\,\mathrm{e}^{-\tau} + T_{\rm ex}(1 - \mathrm{e}^{-\tau})$.

(a) In a population-inverted medium both $\tau < 0$ and $T_{\rm ex} < 0$. Show that for $|\tau| \gg 1$ $T_{\rm b} \approx [T_{\rm b}(0) + |T_{\rm ex}|]\,\mathrm{e}^{|\tau|}$, and say what limits $T_{\rm b}$ in practice.

(b) A seed of $T_{\rm b}(0) = 10^8$ K passes through a maser region with $|T_{\rm ex}| = 10^8$ K. What $|\tau|$ is needed to reach $T_{\rm b} = 10^{15}$ K?

(c) Why can an incoherent (thermal or non-thermal) source never exceed $T_{\rm eff} \sim E/k_{\rm B}$, in terms of τ?

Question 57. A laser pointer versus filtered lamplight

Two 650 nm sources both emit 1 mW through the same 1 mm-diameter aperture. The laser beam has full divergence 1 mrad (take $\Omega = \pi(0.5\,\mathrm{mrad})^2$) and bandwidth 1 GHz; the lamp emits into π sr behind a 10 nm-wide filter. Use $I_\nu = P/(A\,\Omega\,\Delta\nu)$ and $T_{\rm b} = (h\nu/k_{\rm B})/\ln[1 + 2h\nu^3/(c^2 I_\nu)]$.

(a) Calculate ν and $\Delta\nu$ for the lamp ($\Delta\nu = c\,\Delta\lambda/\lambda^2$).

(b) Calculate I_ν and $T_{\rm b}$ for each source. Express the ratio of their I_ν as a product of a solid-angle and a bandwidth factor.

(c) What does this show about thermal light and about the origin of very high $T_{\rm b}$ in the sky?

Question 58. Jupiter: altitude, loss cone and the density condition

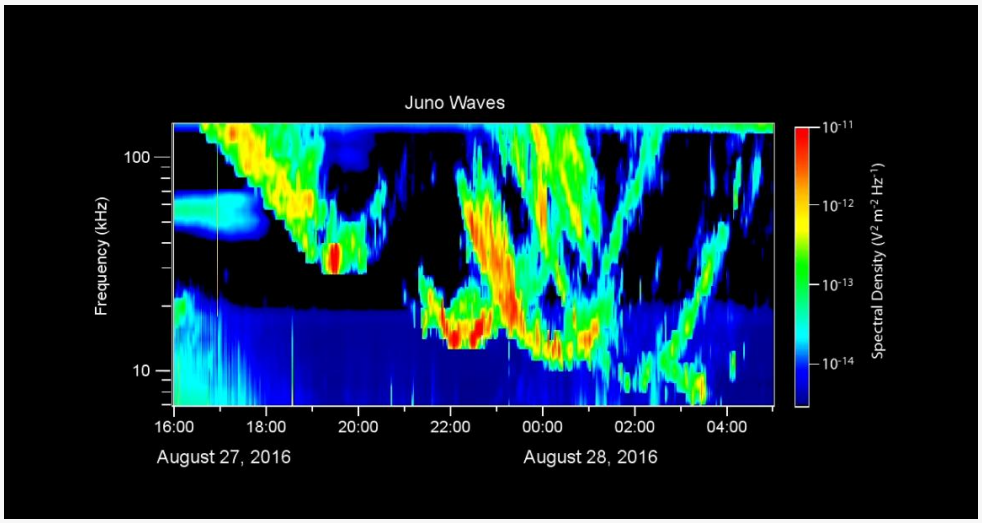
Assume emission at the local gyrofrequency $\nu_{\rm G} = 2.80\,\mathrm{MHz}(B/\mathrm{G})$ (as in Question 54), a dipole field $B \propto r^{-3}$, and a polar surface field of 14 G.

(a) Find the maximum emission frequency, and the distances (in $R_{\rm J}$) of the sources of 1 MHz (hectometric) and 100 kHz (kilometric) emission.

(b) Electrons are mirrored when their pitch angle exceeds $\alpha_{\rm lc} = \sin^{-1}\sqrt{B_{\rm src}/B_{\rm atm}}$. Find the loss-cone half-angle for a source at $3.4R_{\rm J}$ and say why an empty loss cone gives maser action.

(c) The ECM condition is $\nu_{\rm p} \ll \nu_{\rm G}$, $\nu_{\rm p} = 8.98\,\mathrm{kHz}(n_{\rm e}/\mathrm{cm}^{-3})^{1/2}$. Check it for Jupiter's source region ($n_{\rm e} \sim 1\,\mathrm{cm}^{-3}$ at $\nu \sim 1\,\mathrm{MHz}$) and for a solar active-region corona with $n_{\rm e} = 10^{10}\,\mathrm{cm}^{-3}$ and $B = 300\,\mathrm{G}$.

Question 59. Reading the Juno dynamic spectrum



Juno Waves dynamic spectrum of Jupiter's auroral radio emission, 7–140 kHz, 27–28 August 2016 (NASA/JPL-Caltech/SwRI/Univ. of Iowa).

Assume fundamental ECM emission, $\nu = 2.80\,\mathrm{MHz}\,(B/\mathrm{G})$, and a dipole with 40 MHz at the surface ($r = R_\mathrm{J}$).

(a) Read the frequency at the top of the display, and the frequency of the brightest (red) patch near 19:30 on the left.

(b) Calculate the minimum field needed at each of the two frequencies and the distance r/R_J of the source.

(c) The arcs drift with time. Give one reason why this does *not* prove that the source moves, and say why a ground-based telescope could not have recorded this spectrum.

Question 60. Signatures of the cyclotron maser

(a) List four observational signatures of electron-cyclotron maser emission and explain in one sentence each the physical reason for each.

(b) State the conditions (plasma, electron distribution) for ECM to operate, and the condition that it can be *observed* from the ground for an exoplanet. Calculate the minimum surface field for emission to exceed 10 MHz, and compare with an Earth-like field of 0.5 G.

Lecture 10: Spectral-Line Radiative Transfer

Question 61. Line opacity from the Einstein relations

A transition between a lower level l and an upper level u (statistical weights g_l, g_u, level spacing $h\nu$) has a normalised line profile $\phi(\nu)$, $\int \phi\,\mathrm{d}\nu = 1$. The absorption coefficient corrected for stimulated emission is $\kappa_\nu = \dfrac{h\nu}{4\pi}\,(n_l B_{lu} - n_u B_{ul})\,\phi(\nu)$.

(a) Using the Einstein relations $g_l B_{lu} = g_u B_{ul}$ and $A_{ul} = (2h\nu^3/c^2)B_{ul}$, and the definition of the excitation temperature $n_u/n_l = (g_u/g_l)\,e^{-h\nu/k_\mathrm{B}T_\mathrm{ex}}$, show that the optical depth through a column $N_u = \int n_u\,ds$ is

$$\tau_\nu = \frac{c^2 A_{ul}}{8\pi\nu^2}\,N_u\left[e^{h\nu/k_\mathrm{B}T_\mathrm{ex}} - 1\right]\phi(\nu).$$

(b) Take the limit $h\nu \ll k_\mathrm{B}T_\mathrm{ex}$ and state how τ_ν then scales with T_ex for fixed N_u.

Question 62. How much absorption survives stimulated emission?

(a) Using the bracket in the result of Question 61, the fraction of absorbed photons that is *not* returned by stimulated emission is $1 - e^{-h\nu/k_{\rm B}T_{\rm ex}}$. Evaluate it for H I ($\nu = 1420.4\,\mathrm{MHz}$, $T_{\rm ex} = 100\,\mathrm{K}$), NH_3 ($\nu = 23.694\,\mathrm{GHz}$, $T_{\rm ex} = 20\,\mathrm{K}$) and CO 1–0 ($\nu = 115.27\,\mathrm{GHz}$, $T_{\rm ex} = 20\,\mathrm{K}$).

(b) For each line, by what percentage does the approximation $1-e^{-x} \approx x$ (with $x = h\nu/k_{\rm B}T_{\rm ex}$) overestimate the exact value?

(c) An H I cloud has fixed N_u. Its spin temperature is raised from 100 K to 8000 K. By what factor does the peak optical depth change, and why does it matter for absorption surveys?

Use cgs: $h = 6.626 \times 10^{-27}$ erg s, $k_{\rm B} = 1.381 \times 10^{-16}$ erg K^{-1}.

Question 63. Column density from an optically thin line

For an optically thin line with $T_{\rm ex} \gg T_{\rm bg}$, every spontaneous photon escapes, and the upper-level column density is

$$N_u = \frac{8\pi k_{\rm B}\nu^2}{hc^3 A_{ul}} \int T_{\rm b}\,\mathrm{d}v.$$

(a) Starting from Question 61, show how this expression follows for $\tau \ll 1$ and $T_{\rm ex} \gg T_{\rm bg}$ (hint: $T_{\rm b} = T_{\rm ex}\tau$ and $\int \tau\,\mathrm{d}v = c\int \tau\,\mathrm{d}\nu/\nu$).

(b) For H I ($\nu = 1420.406$ MHz, $A_{10} = 2.869 \times 10^{-15}$ s^{-1}, $g_1 = 3$, $g_0 = 1$) with $T_{\rm s} \gg T_* = 0.0682$ K, find the fraction of atoms in the upper level and hence show that $N_{\rm HI} = 1.82 \times 10^{18}\,\mathrm{cm^{-2}} \int T_{\rm b}\,\mathrm{d}v$ with $T_{\rm b}$ in K and v in km s^{-1}.

(c) Compute $N_{\rm HI}$ for $\int T_{\rm b}\,\mathrm{d}v = 150$ K km s^{-1}.

Question 64. Critical density and subthermal excitation

(a) Define the critical density of a transition and explain what happens to $T_{\rm ex}$ for $n \gg n_{\rm crit}$ and for $n \lesssim n_{\rm crit}$.

(b) The CO 1–0 line has $A_{ul} = 7.4 \times 10^{-8}$ s^{-1} and a collisional de-excitation rate coefficient $\gamma_{ul} = 3.3 \times 10^{-11}$ cm^3 s^{-1}. Calculate $n_{\rm crit}$.

(c) A cloud has $n = 300$ cm^{-3} and a line optical depth $\tau = 10$. Using the rule that photon trapping lowers the effective excitation density to about $n_{\rm crit}/\tau$, decide whether the line can still be well excited, and explain why detecting a line does not prove that $n \geq n_{\rm crit}$.

Question 65. Thermal and non-thermal line widths

(a) For particles of mass m with a Maxwellian velocity distribution at temperature $T_{\rm k}$, the line-of-sight velocity is Gaussian with variance $\sigma_v^2 = k_{\rm B}T_{\rm k}/m$. Show that the FWHM is $\Delta v_{\rm th} = \sqrt{8\ln 2\, k_{\rm B}T_{\rm k}/m}$ and that this equals $0.214\,(T_{\rm k}/\mathrm{K})^{1/2}(m/\mathrm{u})^{-1/2}$ km s^{-1}.

(b) NH_3 ($m = 17$ u) lines from a dark cloud have $T_{\rm k} = 15$ K and an observed FWHM of 0.55 km s^{-1}. Find the thermal and the non-thermal contributions, assuming they add in quadrature.

(c) The NH_3 (1,1) line is observed at 23.694 GHz. A width needs at least three resolution elements across the FWHM. What is the largest channel width in kHz that is acceptable? Does a 100-kHz channel suffice?

Use $c = 2.998 \times 10^5$ km s^{-1}, $1\,\mathrm{u} = 1.6605 \times 10^{-24}$ g.

Question 66. Doppler conventions and channel widths

An H I line (rest frequency $\nu_0 = 1420.406\,\mathrm{MHz}$) from a galaxy is observed at $\nu = 1350.0\,\mathrm{MHz}$.

(a) Compute the velocity in the radio convention $c(\nu_0 - \nu)/\nu_0$, the optical convention $c(\nu_0 - \nu)/\nu$, and the relativistic velocity (from $\nu/\nu_0 = \sqrt{(1-\beta)/(1+\beta)}$).

(b) Why does the choice matter here, and which convention gives equal velocity channels for

equal frequency channels?

(c) The correlator has channels 24.414 kHz wide. Find the channel width in $\mathrm{km\,s^{-1}}$ at the observed frequency.

Use $c = 2.998 \times 10^5$ $\mathrm{km\,s^{-1}}$.

Question 67. When the standard assumptions fail

(a) For each of the following assumptions, state in one sentence a physical condition under which it fails and the observational symptom: (i) LTE, $T_{\rm ex} = T_{\rm k}$; (ii) optically thin line; (iii) a single $T_{\rm ex}$ along the sightline; (iv) full beam filling.

(b) Two H I spectra of the same cloud taken six months apart, plotted in *topocentric* velocity, show peaks 20 $\mathrm{km\,s^{-1}}$ apart. Compute the Earth's orbital speed (1 AU $= 1.496 \times 10^{13}$ cm, 1 yr $= 3.156 \times 10^7$ s) and decide whether the cloud need have changed.

Lecture 11: The H I 21-cm Line: Spin Temperature, Phases and H I Mass

Question 68. H I optical depth and column density

Consider a uniform cloud of atomic hydrogen with spin temperature $T_{\rm s} \gg T_* = h\nu/k_{\rm B} = 0.0682$ K, hydrogen density $n_{\rm H}$ and line profile $\phi(\nu)$, $\int \phi \,\mathrm{d}\nu = 1$.

(a) Using $\kappa_\nu = \dfrac{c^2}{8\pi\nu^2}\dfrac{g_1}{g_0}\, n_0 A_{10}\left[1 - e^{-T_*/T_{\rm s}}\right]\phi(\nu)$, $g_1/g_0 = 3$ and $n_0 = n_{\rm H}/4$, show that

$$\int \tau \,\mathrm{d}v = \frac{3c^3 h A_{10}}{32\pi k_{\rm B}\nu^2}\,\frac{N_{\rm HI}}{T_{\rm s}}.$$

(b) Evaluate the numerical coefficient and show that $N_{\rm HI} = 1.82 \times 10^{18}\,\mathrm{cm^{-2}} \int (T_{\rm s}/\mathrm{K})\,\tau \,\mathrm{d}v/(\mathrm{km\,s^{-1}})$.

Use cgs, $\nu = 1420.406$ MHz, $A_{10} = 2.869 \times 10^{-15}$ $\mathrm{s^{-1}}$.

Question 69. Why absorption selects the cold gas

Two H I clouds have the same column density $N_{\rm HI} = 3 \times 10^{20}$ $\mathrm{cm^{-2}}$ and Gaussian line profiles. Cloud C (cold) has $T_{\rm s} = 80$ K and FWHM $= 3$ $\mathrm{km\,s^{-1}}$; cloud W (warm) has $T_{\rm s} = 6000$ K and FWHM $= 22$ $\mathrm{km\,s^{-1}}$.

(a) Using Question 68, find $\int \tau \,\mathrm{d}v$ and the peak optical depth of each cloud. (For a Gaussian, $\int \tau \,\mathrm{d}v = 1.064\,\tau_0\,\Delta v_{\rm FWHM}$.)

(b) Find the peak emission brightness temperature $T_{\rm s}(1 - e^{-\tau_0})$ of each cloud, with no continuum background.

(c) Which cloud dominates the emission, which the absorption against a bright continuum source, and why does this make absorption spectra a census of cold gas?

Question 70. Saturated lines and the thin-limit error

For a uniform slab, $T_{\rm b}(v) = T_{\rm s}\left[1 - e^{-\tau(v)}\right]$ (no background).

(a) Show that the optically thin formula $N_{\rm HI} \propto \int T_{\rm b}\,\mathrm{d}v$ underestimates the true column at a given velocity by the factor $\tau/(1 - e^{-\tau})$.

(b) Evaluate this factor for $\tau = 0.1,\ 0.3,\ 1,\ 3$.

(c) A Galactic-plane spectrum has a flat top with $T_{\rm b} = 114$ K over a range of velocities (the line is at 95 per cent of the plateau maximum, here $T_{\rm s} = 120$ K). Find the optical depth there, and say what can and cannot be concluded about $T_{\rm s}$ and $N_{\rm HI}$ from a plateau.

Question 71. The on–off method for the spin temperature

A compact continuum source with $T_c = 200$ K lies behind an H I cloud. At the cloud's line centre, the spectrum beside the source ("off") shows $T_{\rm off} = 38.0$ K, and the continuum-subtracted spectrum toward the source ("on") shows $\Delta T_{\rm on} = -88.5$ K.

(a) Starting from $T_{\rm off} = T_s(1 - e^{-\tau})$ and $\Delta T_{\rm on} = (T_s - T_c)(1 - e^{-\tau})$, derive expressions for $1 - e^{-\tau}$ and T_s.

(b) Evaluate τ and T_s.

(c) If the line is Gaussian with FWHM 4.0 km s^{-1} and the same τ_0 and T_s apply at line centre, use Question 68 to estimate $N_{\rm HI}$.

(d) State two assumptions of the method.

Question 72. Why gas at 8000 K stays neutral

(a) Evaluate the Saha factor $(2\pi m_e k_B T/h^2)^{3/2} e^{-13.6\,{\rm eV}/k_B T}$ at $T = 8000$ K (cgs), and for a warm neutral medium with total hydrogen density $n_{\rm H} = 0.5$ cm^{-3} solve $x^2/(1-x) =$ (Saha factor)$/n_{\rm H}$ for the LTE ionisation fraction x.

(b) Instead balance collisional ionisation $C n_e n_{\rm H}$ against radiative recombination $\alpha_B n_e n_p$ and show that the ionisation fraction obeys $x/(1-x) = C/\alpha_B$, independent of density. Evaluate it for $C = 10^{-17}$ and $\alpha_B = 3 \times 10^{-13}$ cm^3 s^{-1}.

(c) Why does Saha give the wrong answer for the interstellar medium?

Use $m_e = 9.109 \times 10^{-28}$ g, $1\,{\rm eV} = 1.602 \times 10^{-12}$ erg, $k_B = 1.381 \times 10^{-16}$ erg K^{-1}, $h = 6.626 \times 10^{-27}$ erg s.

Question 73. H I mass from the integrated flux

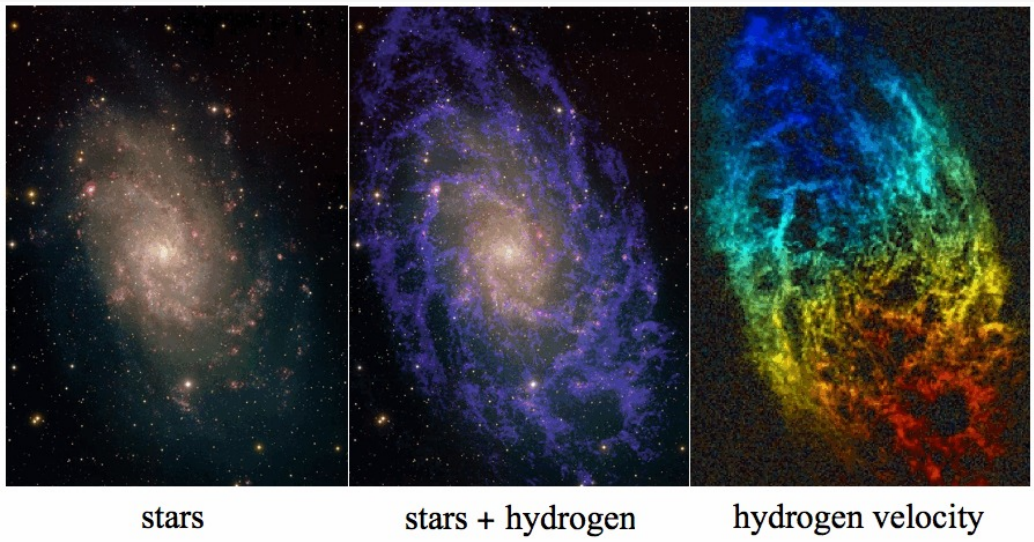
For an optically thin source at distance D (with $z \ll 1$):

(a) Starting from $M_{\rm HI} = m_{\rm H} D^2 \int N_{\rm HI}\, {\rm d}\Omega$, the thin-limit column $N_{\rm HI} = 1.82 \times 10^{18} \int T_b\, {\rm d}v$ and the Rayleigh–Jeans relation $S_\nu = (2k_B\nu^2/c^2) \int T_b\, {\rm d}\Omega$, derive $M_{\rm HI}/M_\odot = 2.35 \times 10^5\, (D/{\rm Mpc})^2 \int S_\nu\, {\rm d}v/({\rm Jy\,km\,s^{-1}})$. (Use $m_{\rm H} = 1.67 \times 10^{-24}$ g, 1 Mpc $= 3.086 \times 10^{24}$ cm, $M_\odot = 1.989 \times 10^{33}$ g; the slide's 2.36×10^5 differs only through rounding of constants.)

(b) A galaxy at 25 Mpc has $\int S_\nu\, {\rm d}v = 12$ Jy km s^{-1}. Find $M_{\rm HI}$ and the total atomic gas mass including helium (factor 1.36).

(c) The distance is uncertain by 20 per cent. What is the fractional uncertainty in $M_{\rm HI}$?

Question 74. Reading an H I image of a galaxy



Nearby spiral galaxy: stars (left), stars plus H I (centre), H I velocity field (right; blue = approaching, red =

receding relative to the systemic velocity).

(a) Compare the extent of the H I with that of the starlight. What does the centre panel show?
(b) What quantity does each pixel of the right-hand panel represent, how is it obtained from the H I data cube $T_{\rm b}(l, b, v)$, and what does the top-to-bottom colour gradient tell you about the galaxy?
(c) Name two reasons why a mass derived from the integrated H I flux of an edge-on, dense disc could be wrong, and the direction of each error.

Lecture 12: H I Stacking and Masers

Question 75. How faint is H I at $z \approx 1$?

A star-forming galaxy at $z = 1.08$ (luminosity distance $D_{\rm L} = 7.5$ Gpc) has $M_{\rm HI} = 1.5 \times 10^{10}\,M_\odot$. For a redshifted source the mass formula of Question 73 becomes $M_{\rm HI}/M_\odot = 2.36 \times 10^5\,(D_{\rm L}/{\rm Mpc})^2\,(1+z)^{-1} \int S_\nu\,{\rm d}v/({\rm Jy\,km\,s^{-1}})$.

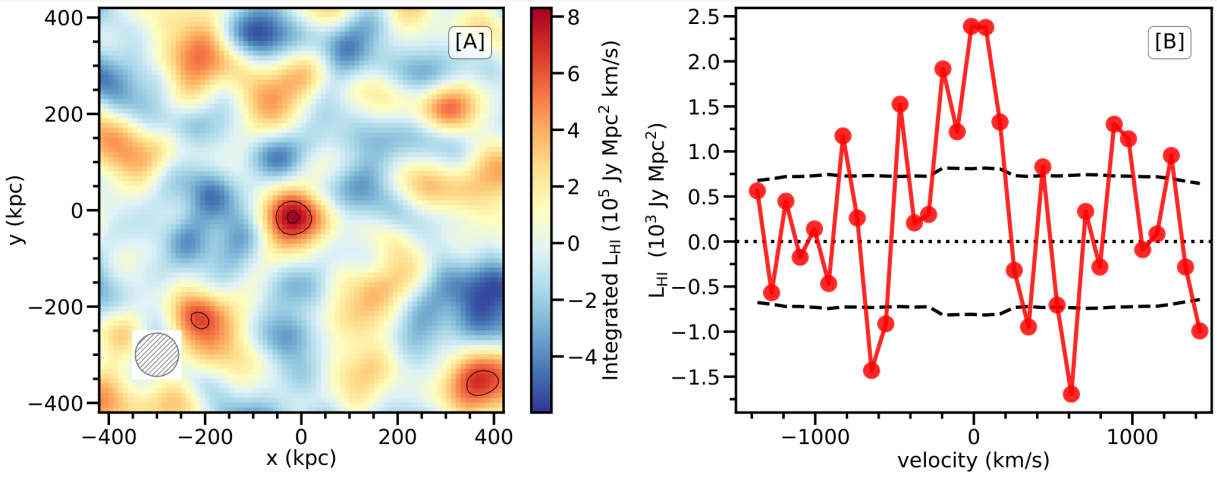
(a) Find the velocity-integrated flux $\int S_\nu\,{\rm d}v$.
(b) The line has a width of 280 $\rm km\,s^{-1}$. Assuming a flat-topped profile, find the flux density S_ν in μJy.
(c) Compare with a channel rms noise of 0.35 mJy and explain why a single-galaxy detection is hopeless.

Question 76. The statistics of stacking

Continue with the signal found in Question 75, now stacking $N = 8122$ galaxies. Take the line observed at $\nu = 1420.406/(1+z)$ MHz, channels of 24 kHz, and an independent noise of $\sigma = 0.35$ mJy per channel per galaxy.

(a) Convert the channel width to $\rm km\,s^{-1}$ and find the number of channels across the 280 $\rm km\,s^{-1}$ line.
(b) Assuming noise that averages down as $1/\sqrt{N}$ and independent between channels, find the rms noise in $\int S_\nu\,{\rm d}v$ and the ideal signal-to-noise ratio of the stacked detection.
(c) The published detection is only 4.2σ. Give two reasons why a real stack does better or worse than this ideal figure, and state what a stacked $M_{\rm HI}$ represents.

Question 77. Reading a stacked H I spectrum



Stacked H I from 8122 galaxies at $z \approx 1$ (Kaur et al. 2026). [A] Integrated $L_{\rm HI}$ map over ± 200 $\rm km\,s^{-1}$ (contours: 3, 4σ). [B] Stacked spectrum in 90 $\rm km\,s^{-1}$ channels (vertical axis in units of 10^3 Jy Mpc^2); the dashed curves are

$\pm1\sigma$.

(a) From panel [B], read the values in the five channels within ±200 km s^{-1} of zero velocity and sum to estimate the integrated $L_{\rm HI}$ in Jy Mpc^2 km s^{-1} (channel width 90 km s^{-1}).
(b) Use $M_{\rm HI} = (2.36\times10^5/4\pi)\,L_{\rm HI}\,M_\odot$ to find $\langle M_{\rm HI}\rangle$. With a mean star-formation rate of $8.9\,M_\odot\,{\rm yr}^{-1}$, find the depletion time $M_{\rm HI}/{\rm SFR}$.
(c) Using $L_{\rm HI} = 4\pi D_{\rm L}^2 \int S_\nu\,{\rm d}v/(1+z)$ with $D_{\rm L} = 7.5$ Gpc, $z = 1.08$, check that your $L_{\rm HI}$ is consistent with the flux of Question 75.
(d) In panel [A], why do the two weaker 3σ peaks away from the centre matter for interpreting the central 4σ peak?

Question 78. Maser amplification and negative excitation temperature

(a) Starting from $n_{\rm u}/n_{\rm l} = (g_{\rm u}/g_{\rm l})\,e^{-h\nu/k_{\rm B}T_{\rm ex}}$, show that a population inversion $n_{\rm u}/g_{\rm u} > n_{\rm l}/g_{\rm l}$ implies $T_{\rm ex} < 0$.
(b) For the 22.235 GHz H_2O line, the inversion is $(n_{\rm u}/g_{\rm u})/(n_{\rm l}/g_{\rm l}) = 1.001$. Find $T_{\rm ex}$.
(c) The intensity obeys $I_\nu \approx I_{\nu,0}\,e^{|\tau|}$ with $|\tau| = 20$. Find the gain, and the output brightness temperature for a seed of $T_{\rm b} = 100$ K (use $T_{\rm b} = T_{\rm ex}(1-e^{-\tau}) + T_{\rm bg}e^{-\tau}$ with $T_{\rm bg} = 100$ K and $\tau = -20$).
(d) Explain why this does not violate the thermal limit $T_{\rm b} \le T_{\rm k}$, and what happens when the maser saturates.

Use cgs: $h = 6.626\times10^{-27}$ erg s, $k_{\rm B} = 1.381\times10^{-16}$ erg K^{-1}.

Question 79. What masers are, and are not, good for

(a) List the three requirements for a maser and, in one sentence each, the observable consequence of exponential gain in a compact region (name at least three observational properties).
(b) Why is maser flux density *not* a measure of the mass or column density of the emitting gas?
(c) Name the observables that masers do measure well, and give one Galactic and one extragalactic application.

Question 80. Maser parallaxes and Galactic kinematics

(a) A CH_3OH maser at 6.7 GHz has a VLBI parallax $\pi = 0.125 \pm 0.010$ mas. Find its distance and the uncertainty.
(b) The same source has a proper motion $\mu = 3.0$ mas yr^{-1}. Using 1 AU yr^{-1} = 4.74 km s^{-1}, derive and evaluate the transverse velocity $v_\perp = 4.74\,\mu\,d$ (with μ in mas yr^{-1}, d in kpc).
(c) Why can only masers, and not thermal lines, provide such parallaxes?

Use 1 AU $= 1.496\times10^{13}$ cm, 1 yr $= 3.156\times10^7$ s.

Question 81. A geometric distance to a megamaser disc

Edge-on-ish discs of H_2O masers orbit a black hole. For systemic masers at orbital radius r and orbital speed v (Keplerian), the line-of-sight acceleration is $a = (v^2/r)\sin i$, where i is the disc inclination from the line of sight.

(a) Show that the distance is $D = v^2 \sin i/(a\,\theta)$, where θ is the angular radius measured with VLBI.
(b) Take $a = 9.0$ km s^{-1} yr^{-1}, $v = 1120$ km s^{-1}, $i = 71.7^\circ$, $\theta = 3.7$ mas. Find r (in pc) and D (in Mpc).
(c) Using a, v and θ with fractional errors 3, 1 and 1 per cent, estimate the fractional error on D.
(d) Why does this method need no distance ladder?

Use 1 mas $= 4.848 \times 10^{-9}$ rad, 1 pc $= 3.086 \times 10^{16}$ m, 1 yr $= 3.156 \times 10^{7}$ s.