The Radio Sky: Problem Sheet 4
IUCAA-NCRA Graduate School

October – December 2025

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- These problems are for your own practice and will not be graded. They are designed to help you prepare for the mid-term and final examinations. However, I strongly encourage you to ask questions and discuss the solutions.
- If you spot any potential errors or find a question unclear, please do not hesitate to let me know.
- You are welcome to consult books, online resources, and discuss the problems with your peers. The key, however, is to ensure you personally understand the solutions, as this will be vital for your performance in the examinations.
- If you choose to use notation or conventions that differ from those presented in lectures, please define them clearly at the start and apply them consistently.

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1. Consider the solar corona as an isothermal plasma at temperature $T_c = 10^6$ K. The electron density falls off radially as a power law

$$n_e(r) = n_0 \left(\frac{r}{R_\odot}\right)^{-\beta},$$

where r is the distance from the centre of the Sun, n_0 is the density at the base of the corona ($r = R_\odot$), and $\beta > 1.5$. Assume the observer is located at a distance $D \gg R_\odot$, such that lines of sight passing through the corona can be treated as parallel rays parameterized by their impact parameter b.

The free-free absorption coefficient is approximated by $\kappa_\nu \approx A\, n_e^2(r)\, \nu^{-2}\, T^{-3/2}$.

   (a) Set up the integral for the optical depth $\tau_\nu(b)$ along a line of sight with impact parameter b (where $b > R_\odot$).

   (b) Find the dependence of $\tau_\nu(b)$ on b and ν. For a given frequency ν, does the corona become optically thin at larger or smaller impact parameters?

   (c) Show that the apparent size of the Sun scales as a power law of frequency $R_\nu \propto \nu^{-\gamma}$. Determine the index γ in terms of the density power-law index β.

2. Consider a star in which gas pressure and radiation pressure are both important (i.e., the total pressure is the sum of the two).

   (a) If the gas pressure is equal to a constant fraction β of the total pressure everywhere inside the star, then show that the total pressure has to be related to the density in the following way

$$P = \left(\frac{3k_B^4}{a_B \mu^4 m_p^4}\right)^{1/3} \left(\frac{1-\beta}{\beta^4}\right)^{1/3} \rho^{4/3},$$

   where a_B is the radiation constant, k_B is the Boltzmann constant, m_p is the proton mass, and μ is the mean molecular weight.

(b) Now consider several stars with different masses having the same composition (i.e. the same μ). Assuming that inside each of these stars the gas pressure is everywhere a constant fraction β of the total pressure (but β has different values for different stars), show that β inside a star would be related to its mass M by an equation of the form

$$\frac{1-\beta}{\beta^4} = CM^2,$$

where C is a constant which you have to evaluate. Show that β is smaller for larger M, implying that radiation pressure is increasingly more important inside more massive stars. This is a historically important argument first given by Eddington.

(c) Calculate the value of $M/M_\odot$ for which $\beta = 0.5$ (i.e., the radiation pressure is equal to the gas pressure). Also calculate the value of β for the Sun. Assume $\mu \approx 0.62$ (appropriate for solar composition).

3. Let us model a disk galaxy as an infinitely thin circular disk having a surface density $\Sigma(R)$.

(a) Argue that the density of the disk can be written as $\rho(R, z) = \Sigma(R)\, \delta_D(z)$.

(b) Show that the most general solution of the corresponding Poisson equation can be written as

$$\Phi(R, z) = \int_0^\infty \mathrm{d}k\; S(k)\; \mathrm{e}^{-k|z|} J_0(kR)$$

Write the function $S(k)$ in terms of $\Sigma(R)$.

(c) If we choose a form of $\Sigma(R)$ appropriate for spiral disk galaxies

$$\Sigma(R) = \Sigma_0\, \mathrm{e}^{-R/R_d}$$

show that

$$\Phi(R, z) = -2\pi G\, \Sigma_0\, R_d^2 \int_0^\infty \frac{\mathrm{d}k}{(1 + k^2 R_d^2)^{3/2}} \mathrm{e}^{-k|z|} J_0(kR)$$

(d) Evaluate the integral for $z = 0$ and show that it is given by

$$\Phi(R, 0) = -\pi G\, \Sigma_0 R \left[I_0(y) K_1(y) - I_1(y) K_0(y) \right]$$

where

$$y \equiv \frac{R}{2R_d}$$

(e) Evaluate the corresponding circular speed. What are its limiting forms for $R \ll R_d$ and $R \gg R_d$?

4. Consider an O-type star driving a stellar wind with a mass-loss rate $\dot{M}$ and terminal velocity v_∞. At the same time, the star also emits ionizing photons at a rate Q_*.

(a) Use the mass conservation to show that the wind density profile is $n_e(r) = Ar^{-2}$. Assume the wind is composed purely of ionized hydrogen, so that $n_e = n_p = n_H$.

(b) Derive the condition for the wind to be optically thick to the star's own ionizing radiation. That is, find the critical mass-loss rate $\dot{M}_{crit}$ (in terms of v_∞, Q_*, and recombination coefficient α_B) above which the entire Strömgren volume is contained within the wind itself.

(c) If the wind is optically thick to ionizing photons ($\dot{M} > \dot{M}_{crit}$), show that the radio spectral index α (where $F_\nu \propto \nu^\alpha$) is $\alpha \approx 0.6$.

(d) If the wind is optically thin ($\dot{M} \ll \dot{M}_{crit}$), the ionizing photons escape the wind and ionize the surrounding static ISM, creating a classical Strömgren sphere. Sketch the expected radio spectrum F_ν vs ν (log-log) for this composite system, clearly labelling the turnover frequencies and spectral indices for both the wind component and the static HII region component.

5. We observe a *repeating* Fast Radio Burst (FRB) source at a cosmological distance. The bursts have the following properties:

   - Isotropic energy per burst: $E_{\rm burst} \approx 10^{40}$ erg.
   - Burst duration: $W \approx 1$ ms.
   - Repetition Rate: One burst every $\Delta t \approx 10$ minutes.

   We consider two candidate engines for this source, both involving a neutron star ($R = 10$ km, $I = 10^{45}$ g cm^2):

   - A young, fast rotation-powered radio pulsar with period $P = 10$ ms and surface magnetic field $B = 10^{12}$ G. The energy source is the rotational kinetic energy ($E_{\rm rot}$) of the star.
   - A magnetar with period $P = 5$ s and surface magnetic field $B = 10^{15}$ G. The energy source is the decay of the internal magnetic field ($E_{\rm mag}$).

   (a) Calculate the time-averaged luminosity $\langle L \rangle$ required to sustain the observed repetition rate.

   (b) Calculate the total energy reservoir available in rotation-powered model ($E_{\rm rot} = I\Omega^2/2$) and magnetically-powered model ($E_{\rm mag} \approx (B^2/8\pi)V_{\rm NS}$). Show that both models possess sufficient total energy to sustain this activity for at least 100 years.

   (c) Calculate the peak luminosity $L_{\rm peak}$ of a single millisecond burst. For the rotation-powered model, the power is extracted via magnetic dipole braking. The maximum rate at which rotational energy can be converted to radiation is the spin-down luminosity

   $$\dot{E}_{\rm rot} \approx \frac{B^2 R^6 \Omega^4}{c^3},$$

   assuming the internal and surface magnetic fields are comparable. Calculate $\dot{E}_{\rm rot}$ for this model. Can this standard pulsar mechanism explain the FRB peak luminosity?

   For the magnetically-powered model, energy is released via magnetic reconnection (similar to a solar flare). Explain qualitatively why this mechanism is not limited by a steady "drain" rate like rotation, and how it can resolve the timescale issue found in the rotation-powered model.

   (d) Based on your results from the above parts, which engine is the physically favoured candidate for cosmological FRBs, and why?

6. Consider a Quasar with a bolometric luminosity of $L_{\rm bol} = 10^{47}$ erg s^{-1}.

   (a) Calculate the mass accretion rate $\dot{M}_{\rm BH}$ (in $M_\odot$ yr^{-1}) required to sustain this luminosity, assuming a standard radiative efficiency for a non-rotating black hole ($\epsilon_{\rm acc} \approx 0.08$).

   (b) Now, consider a Starburst galaxy with the same total bolometric luminosity, powered entirely by high-mass star formation. Assuming the Salpeter Initial Mass Function implies that only $\sim 10\%$ of the mass formed ends up in massive stars that fuse hydrogen efficiently, estimate the Star Formation Rate (SFR) in $M_\odot$ yr^{-1} required to match the luminosity of the quasar.

   (c) A typical massive spiral galaxy has a total cold gas reservoir (HI + H_2) of $M_{\rm gas} \approx 10^{10} M_\odot$. Using your SFR from previous part, calculate the gas depletion timescale ($t_{\rm dep} = M_{\rm gas}/{\rm SFR}$). Compare $t_{\rm dep}$ to the dynamical timescale of a galaxy ($\tau_{\rm dyn} \sim 200$ Myr). What does this imply about the nature of such high-luminosity starbursts? Are they long-lived equilibrium states or transient events?

7. A radio source exhibits a one-sided jet. Long-term VLBI monitoring reveals that the jet components move with an apparent transverse velocity $\beta_{\rm app} = 6$ ($v_{\rm app} = 6c$). The jet has a continuous flow with spectral index $\alpha = -0.7$. We rarely have perfect knowledge of the intrinsic velocity β, so astronomers often assume the "minimum velocity solution", the lowest physical β capable of producing the observed apparent speed.

   (a) Derive the minimum intrinsic velocity $\beta_{\rm min}$ (and corresponding Lorentz factor $\gamma_{\rm min}$) required to produce $\beta_{\rm app} = 6$.

   (b) Calculate the viewing angle θ corresponding to this minimum velocity solution.

(c) Using the parameters from the previous parts, calculate the expected brightness ratio R between the approaching jet and the receding counter-jet.

(d) Typical radio images have a dynamic range (contrast) of about $1000 : 1$. Based on your calculation, explain why counter-jets are almost never detected in superluminal sources.

8. In a static, uniform Euclidean universe, the cumulative number of sources brighter than flux S follows the famous power law $N(> S) \propto S^{-1.5}$. Now, consider a population of "standard candle" radio sources (luminosity L) that undergo pure density evolution. Their number density increases with redshift as

$$\rho(z) = \rho_0(1+z)^k \quad \text{where } k > 0.$$

Assume we are observing in the local universe ($z \ll 1$) where Euclidean geometry still holds for volume ($V = 4\pi r^3/3$) and flux ($S = L/4\pi r^2$).

Using the low-redshift approximation for the Hubble Law, $cz \approx H_0 r$, derive the expression for the cumulative number of sources $N(> S)$. Hence calculate the "effective slope" $\gamma_{\text{eff}} = -\mathrm{d}\ln N/\mathrm{d}\ln S$ in the limit of very bright sources ($S \to \infty$). Does the slope steepen or flatten as you move to fainter fluxes?

9. Consider a particle moving along a geodesic with $\theta =$ constant and $\phi =$ constant in a FRW universe, with the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right].$$

Note that we are working in units where $c = 1$.

(a) Show that the zeroth component of the geodesic equation reads as

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + \frac{a\dot{a}}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = 0$$

(b) Now, use the normalization condition for the four-velocity to show that

$$\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \frac{a^2}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = 1$$

(c) Eliminate $\mathrm{d}r/\mathrm{d}s$ from the two equations to obtain a differential equation for $t(s)$. Then integrate the differential equation and show that the solution is

$$a^2\left[\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - 1\right] = \text{constant}$$

(d) Use the above to show that the magnitude of the three-momentum of the particle varies as a^{-1}.