Quantum Mechanics: Assignment 0

IUCAA-NCRA Graduate School

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- The questions in this assignment are based on standard courses at B.Sc./M.Sc. level.
- You may look up textbooks and/or consult friends for solving the problems, but make sure you understand the solutions.
- You need *not* submit this assignment. However, if you find any of these questions nontrivial/difficult, please let me know so that the rest of the course can be designed appropriately.

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1. **Probability fluid:** Show that the one particle time-dependent Schrödinger equation is equivalent to the set of equations

$$\boldsymbol{\nabla} \times \boldsymbol{v} = 0, \quad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0,$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{m}\boldsymbol{\nabla} V - \frac{\hbar^2}{2m^2}\boldsymbol{\nabla}\left(\frac{\boldsymbol{\nabla}^2 \rho^{1/2}}{\rho^{1/2}}\right),$$

where $\psi = \rho^{1/2}\, \mathrm{e}^{\mathrm{i}S/\hbar}$ and $\boldsymbol{v} = \boldsymbol{\nabla} S/m$. Note that both ρ and S are real.

2. **Properties of the Schrödinger equation:** If two (or more) distinct solutions to the (time-independent) Schrödinger equation have the same energy E, these states are said to be degenerate. Prove that there are no degenerate bound states in one dimension.

3. **Linear operators:**

   (a) Show that the eigenvalues of a Hermitian operator are always real.

   (b) Show that the eigenfunctions belonging to distinct eigenvalues of a Hermitian operator are orthogonal.

   (c) If the operator H is Hermitian, show that $\mathrm{e}^{\mathrm{i}H}$ is unitary. What are the possible eigenvalues of a unitary operator?

4. **One-dimensional potential:** Consider a quantum mechanical particle of mass m moving in the one-dimensional potential

$$\begin{aligned} V(x) &= \infty \quad \text{for } x < 0, \\ &= -V_0 \quad \text{for } 0 \le x \le a, \\ &= 0 \quad \text{for } x > a, \end{aligned}$$

   where $V_0 > 0$, $a > 0$. Assume the particle to be in a stationary state.

   (a) Solve the Schrödinger equation and obtain the wave function for the system for particle energies $-V_0 < E < 0$. What are the allowed values of E?
   Hint: You may have to solve the equations numerically / graphically, in that case find the allowed values of E for $V_0 = (\hbar^2/2m)\ (a/10)^2$.

   (b) Obtain the solutions for $E > 0$. What is the reflection coefficient?

5. **The delta function potential:** A particle of mass m is bound in the one-dimensional delta function well $V(x) = -\alpha\delta_D(x)$. What is the probability that a measurement of its momentum would yield a value greater than $p_0 = m\alpha/\hbar$?

6. **The simple harmonic oscillator:** Consider the one-dimensional simple harmonic oscillator potential $V(x) = m\omega^2 x^2/2$.

   (a) Show that the eigenvalues of the Hamiltonian H of the system *cannot* be negative.

   (b) Show that the Hamiltonian of the system can be written as

$$H = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right),$$

   where the operators $a_\pm$ are defined as

$$a_\pm = \sqrt{\frac{1}{2\hbar m\omega}}\ (m\omega x \mp \mathrm{i}p)\,.$$

(c) Show that if $|\psi\rangle$ is an eigenstate of H with eigenvalue E, then $a_\pm|\psi\rangle$ too are eigenstates of H but with eigenvalues $E \pm \hbar\omega$.

(d) Argue that the ground state $|\psi_0\rangle$ of the system (i.e., the eigenstate with the lowest value of E) must satisfy

$$a_-|\psi_0\rangle = 0.$$

Solve the equation to obtain the normalized ground state. What is the value of the ground state energy?

(e) Apply a_+ successively on the ground state and show that the nth energy eigenstate is given by

$$|\psi_n\rangle = \frac{1}{\sqrt{n!}}\,(a_+)^n\,|\psi_0\rangle.$$

(f) Write down the time-independent Schrödinger equation for a particle moving in the potential, solve it using appropriate boundary conditions and obtain the energy levels as well as the wave functions.

7. **Evolution of a wave packet:** Consider a particle of mass m moving in the harmonic oscillator potential $V(x) = m\omega^2 x^2/2$. Let the wave function at $t = 0$ be given by a gaussian with zero mean and width σ.

   (a) Find the wave function at $t > 0$.

   (b) Show that, for an arbitrary value of σ, the width of the wave function oscillates in time. Also show that it is possible to choose σ such that the width of the wave function does not vary with time.

   (c) Expand the wave function at $t > 0$ in terms of the oscillator stationary states. Hence find the probability of finding the oscillator in the nth energy state at time t.

8. **The spherical harmonic oscillator:** Consider the three-dimensional harmonic oscillator which has a potential

$$V(r) = \frac{1}{2} m\omega^2 r^2.$$

   (a) Show that separation of variables in *Cartesian* coordinates simplifies the system into three one-dimensional oscillators, and hence determine the allowed energy levels. Also determine the degeneracy of the energy levels.

   (b) Because the three-dimensional harmonic oscillator potential is spherically symmetric, the Schrödinger equation can also be solve by separation of variables in *spherical* coordinates. Obtain the solutions to the angular part. Use the power series method to solve the radial equation. Find the recursion formula for the coefficients, and determine the allowed energies as well as the degeneracy.

9. **Fermi oscillator:** Consider a system with the Hamiltonian

$$H = f^\dagger\, f$$

   where f is an operator that satisfies

$$f^2 = 0, \quad \{f^\dagger, f\} = 1.$$

   (a) Show that $H^2 = H$ and thus find the eigenvalues of H.

   (b) Let $|\psi_0\rangle$ be the ground state and $|\psi_1\rangle$ be the (first) excited state if the system. Consider the following states

$$f|\psi_0\rangle, \quad f^\dagger|\psi_0\rangle, \quad f|\psi_1\rangle, \quad f^\dagger|\psi_1\rangle.$$

   Find the norms of these states. Which of the above are eigenstates of H, and what are the corresponding eigenvalues?

10. **Properties of the Pauli matrices:** The Pauli spin matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   Show that

   (a) $[\sigma_i, \sigma_j] = 2\,\mathrm{i}\,\varepsilon_{ijk}\,\sigma_k$, where ε_{ijk} is the Levi-Civita symbol,

   (b) $\{\sigma_i, \sigma_j\} = 2\,\delta_{ij}\,1_2$ where 1_2 is the 2×2 identity matrix,

   (c) $\sigma_i\sigma_j = \delta_{ij}\,1_2 + \mathrm{i}\sum_k \varepsilon_{ijk}\,\sigma_k$,

   (d) $(\boldsymbol{\sigma}\cdot\boldsymbol{a})\,(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\boldsymbol{b})1_2 + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$, where $\boldsymbol{\sigma} = \sigma_1\hat{x} + \sigma_2\hat{y} + \sigma_3\hat{z}$ and $\boldsymbol{a}, \boldsymbol{b}$ are ordinary vectors.