Methods of Mathematical Physics I: Exercise Sheet 2

IUCAA-NCRA Graduate School

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1. **Green's function for a damped harmonic oscillator:** Consider a damped harmonic oscillator driven by an external force $F(t)$:

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = F(t), \qquad t > 0,$$

with the boundary condition $x(0) = \dot{x}(0) = 0$. Note that the damping coefficient γ and the natural frequency ω_0 are independent of t. Also assume that the system is underdamped, i.e.,$\omega_0 > \gamma$.

(i) Find the solution(s) $y_1(t)$ and $y_2(t)$ to the homogeneous equation.

(ii) Write the differential equation satisfied by the Green's function for the system. Assume the impulse to be at $t = s$.

(iii) For $t \neq s$, the Green's function is simply the linear combination of $y_1(t)$ and $y_2(t)$, i.e.,

$$\begin{aligned} G(t,s) &= c_1 y_1(t) + c_2 y_2(t) \qquad x < s, \\ &= d_1 y_1(t) + d_2 y_2(t) \qquad x > s. \end{aligned}$$

Find the constants using the boundary conditions and then write down the Green's function.

2. **Hilbert transforms:** Let the Hilbert transform of a function $u(t)$ b e defined as

$$v(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \mathrm{d}t' \, \frac{u(t')}{t' - t}.$$

Show that the inverse transform is given by

$$u(t) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \mathrm{d}t' \, \frac{v(t')}{t' - t}.$$

Hint: Find the relation between the Fourier transforms of $u(t)$ and $v(t)$.

3. **Linear filter:** A linear system has been supplied a input $f(t)$ of the form

$$f(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \mathrm{e}^{-\lambda t} & \text{for} \quad t > 0 \end{cases}$$

where λ is a fixed positive constant, and the output is observed to be

$$g(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \left(1 - \mathrm{e}^{-\alpha t}\right)\, \mathrm{e}^{-\lambda t} & \text{for} \quad t > 0 \end{cases}$$

where α is another fixed positive constant. Find the transfer function $\tilde{G}(\omega)$ and also find the response of the system to the input $f(t) = \delta(t)$.

Hint: You may need to use the following integral:

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega + \mathrm{i}a} = -2\pi \mathrm{i} \mathrm{e}^{-at}\ \Theta(t)$$

4. **Correlations for the linear filter:** Consider the linear filter having an output

$$g(t) = \int \mathrm{d}\tau \ G(\tau)\ f(t - \tau)$$

where $f(t)$ is the input and $G(\tau)$ is the transfer function. Then show that

$$\mathrm{CCF}(\tau) = \int \mathrm{d}\tau' \ G(\tau')\ \mathrm{ACF}(\tau - \tau')$$

where the correlation functions are defined as

$$\mathrm{ACF}(\tau) = \int \mathrm{d}t \ f(t)\ f(t - \tau); \quad \mathrm{CCF}(\tau) = \int \mathrm{d}t \ g(t)\ f(t - \tau)$$

5. **Klein-Gordon equation:** Consider the time-dependent inhomogeneous Klein-Gordon equation:

$$\left[\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + m^2\right] \phi(t, \boldsymbol{x}) = \rho(t, \boldsymbol{x}).$$

Using Fourier transforms, show that the Green's function is given by

$$G(t, \boldsymbol{x}, t', \boldsymbol{x'}) = \frac{1}{8\pi^2 R} \int_{-\infty}^{\infty} \mathrm{d}\omega \ \mathrm{e}^{-\mathrm{i}\omega(t-t')} \mathrm{e}^{\pm \mathrm{i}\sqrt{\omega^2 - m^2}\ R}$$

where $R = |\boldsymbol{x} - \boldsymbol{x'}|$.

6. **Slits of finite size:** (i) Consider an opaque screen having two one-dimensional slits of width d placed at $\pm a/2$. The slits are illuminated by a normally incident, unit amplitude monochromatic plane wave of wavelength λ. Show that the intensity distribution of radiation at a distance z from the screen is given by

$$I(x, z) = \left(\frac{2d}{\lambda z}\right)^2 \ \mathrm{sinc}^2\left(\frac{\pi x d}{\lambda z}\right) \ \cos^2\left(\frac{\pi x a}{\lambda z}\right)$$

You may assume Fraunhoffer approximation to be valid.

7. **Energy for the wave equation:** Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(i) Define the kinetic and potential energies as

$$K = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x \ \left(\frac{\partial u}{\partial t}\right)^2; \quad P = \frac{c^2}{2} \int_{-\infty}^{\infty} \mathrm{d}x \ \left(\frac{\partial u}{\partial x}\right)^2$$

Show that the total energy is conserved, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}(K + P) = 0$$

(ii) Using the energy conservation or otherwise, show that the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad u(t = 0, x) = u_0(x); \quad \frac{\partial u(t = 0, x)}{\partial t} = v_0(x)$$

has a unique solution.

8. **Gaussian temperature profile:** Consider the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

where κ is the conductivity. Let the initial condition be given by a Gaussian

$$T(t = 0, x) = T_0 \mathrm{e}^{-x^2/a^2}$$

(i) Assume that the temperature profile maintains the Gaussian shape but the parameters may change. Then one can consider a solution of the form

$$T(t, x) = F(t) \ \mathrm{e}^{-H(t) x^2}$$

where $F(t), H(t)$ are to be determined. What are the initial conditions satisfied by these two unknown functions?

(ii) Show that the functions should satisfy the differential equations

$$\frac{\mathrm{d}F}{\mathrm{d}t} = -2\kappa F H; \quad \frac{\mathrm{d}H}{\mathrm{d}t} = -4\kappa H^2$$

(iii) Solve the two ODEs, put the initial conditions and find the temperature profile $T(t, x)$ for all times.

9. **Uniqueness of solutions for diffusion equation:** Consider the one-dimensional diffusion equation whose solution is defined within an interval (a, b):

$$\frac{\partial p(t,x)}{\partial t} = \alpha \frac{\partial^2 p(t,x)}{\partial x^2}, \quad t > 0, \quad a < x < b.$$

The initial condition is $p(t=0,x) = p_0(x)$ and the boundary conditions are given by $p(t, x=a) = f_1(t),\ p(t, x=b) = f_2(t)$. These are known as Dirichlet boundary conditions.

(i) Let us assume the above problem has two solutions p_1 and p_2. Define a new quantity $q(t,x) = p_1(t,x) - p_2(t,x)$. Then show that the $w(t,x)$ too satisfies the diffusion equation with the conditions $q(t=0,x) = 0$ and $q(t,x=a) = q(t,x=b) = 0$.

(ii) If we define

$$E(t) \equiv \int_a^b \mathrm{d}x \; q^2(t,x),$$

where $q(t,x)$ is defined in the previous part, then show that $\mathrm{d}E/\mathrm{d}t \leq 0$.

(iii) Show that according to the initial and boundary conditions $E(t=0) = 0$. Hence argue that $E(t) = 0$ at all times. This proves the uniqueness of solutions.

(iv) Repeat the same problem for Neumann boundary conditions in which case the initial condition remains the same but the boundary conditions are given by $\partial p(t, x=a)/\partial x = g_1(t),\ \partial p(t, x=b)/\partial x = g_2(t)$.

10. **Invariance properties of the diffusion equation:** Show that the diffusion equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

is invariant under the following transformations:

(i) Spatial translations: If $u(t,x)$ is a solution of the diffusion equation, then so is the function $u(t, x - x')$ for any fixed x'.

(ii) Differentiation: If u is a solution of the diffusion equation, then so are $\partial u/\partial x, \partial u/\partial t, \partial u/\partial x^2, \ldots$ and so on.

(iii) Linear combinations: If $u_1, u_2, \ldots, u_n$ are solutions of the diffusion equation, then so is $u = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$ for any constants $c_1, c_2, \ldots, c_n$.

(iv) Integration: If $S(t,x)$ is a solution of the diffusion equation, then so is the integral

$$u(t,x) = \int_{-\infty}^{\infty} \mathrm{d}x' \; S(t, x - x') \; g(x')$$

for any function $g(x')$ (as long as the integral converges).

(v) Dilation (scaling): If $u(t,x)$ is a solution of the diffusion equation, then so is the dilated function $v(t,x) = u(a\,t, \sqrt{a}\,x)$ for any constant $a > 0$.

11. **Solving the general diffusion equation:** Let us consider the system

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}; \quad v(t=0,x) = \Theta(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x < 0 \end{cases}$$

(i) Show that the above system (i.e., both the PDE and the initial condition) is invariant under the scaling $t \to a^2 t, x \to at$. Hence argue that the solution can depend only on the ratio $x/\sqrt{t}$, i.e.,

$$v(t,x) = A\ V\left(\frac{x}{\sqrt{t}}\right) \equiv A\ V(\xi); \quad \xi = \frac{x}{\sqrt{t}}$$

where A is a constant.

(ii) Show that V satisfies the ordinary differential equation

$$V'' + \frac{\xi}{2\alpha} V' = 0$$

Solve the equation and show that

$$v(t,x) = c_1 \int_0^{x/\sqrt{4\alpha t}} \mathrm{d}\eta \; \mathrm{e}^{-\eta^2} + c_2$$

(iii) Put in the initial conditions to determine c_1 and c_2.

(iv) Using the invariance properties of the diffusion equation, show that

$$u(t,x) = \int_{-\infty}^{\infty} \mathrm{d}x' \; S(t, x - x') \; u_0(x')$$

is a solution to the diffusion equation, where

$$S(t,x) = \frac{\partial v(t,x)}{\partial x}$$

and $u_0(x)$ is an arbitrary function.

(v) Calculate the initial value $u(t = 0, x)$. Hence show that $u(t,x)$ is a solution to the general equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}; \quad u(t = 0, x) = u_0(x)$$

(vi) Compute the explicit expression for $S(t,x)$ and show that it is the same propagator derived in the class.