Methods of Mathematical Physics I: Assignment 2

IUCAA-NCRA Graduate School

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12 September 2012

To be returned in the class on 21 September 2012

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Riemann-zeta function:** (i) Expand the function

$$f(x) = x^2, \qquad -\pi < x < \pi$$

in a Fourier series and show that

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}.$$

(ii) Put $x = \pi$ and show that the value of the zeta function $\zeta(2)$ is given by

$$\zeta(2) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[5+2]

2. **Triangular wave:** A triangular wave is represented by

$$f(x) = \begin{cases} x & \text{for} \quad 0 < x < \pi \\ -x & \text{for} \quad -\pi < x < 0. \end{cases}$$

Represent $f(x)$ by a Fourier series.

[5]

3. **Fourier coefficients using minimization techniques:** A function $f(x)$ (assumed to be quadratically integrable) is to be represented by a *finite* Fourier series. A convenient measure of the accuracy of the series is given by the integrated square of the deviation,

$$\Delta_N = \int_{-L/2}^{L/2} \mathrm{d}x \, \left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{N} a_n \cos\left(\frac{2n\pi x}{L}\right) - \sum_{n=1}^{N} b_n \sin\left(\frac{2n\pi x}{L}\right)\right]^2.$$

Show that the requirement that Δ_n be minimized, i.e.,

$$\frac{\partial \Delta_N}{\partial a_n} = 0, \quad \frac{\partial \Delta_N}{\partial b_n} = 0$$

for all n, leads to choosing a_n and b_n as given in standard formulae for the Fourier series.

[10]

4. **Fourier transform of a Bessel function:** Use the integral representation of the Bessel function

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \; \mathrm{e}^{\mathrm{i}x\cos\theta}$$

to show that its Fourier transform can be expressed as

$$\tilde{g}(k) = \int_{-\infty}^{\infty} \mathrm{d}x \; J_0(x) \; \mathrm{e}^{-\mathrm{i}kx} = \int_0^{2\pi} \mathrm{d}\theta \; \delta(k - \cos\theta).$$

Noting that the delta function is never satisfied for $|k| > 1$, and that there are two values of the θ which satisfy it for $|k| < 1$, show that

$$\tilde{g}(k) = \begin{cases} \dfrac{2}{\sqrt{1-k^2}} & \text{for} \quad |k| < 1 \\ 0 & \text{for} \quad |k| > 1. \end{cases}$$

[8]

5. **Fourier transform for even/odd functions:** (i) Suppose the function $f(x)$ is even. Then show that the Fourier transform is given by the cosine transform

$$\tilde{f}(k) = 2 \int_0^{\infty} \mathrm{d}x \; f(x) \; \cos kx.$$

What is the inverse relation?

(ii) Repeat the above problem for the case when $f(x)$ is an odd function.

[3+3]

6. **Properties of Fourier transform:** (i) Show that $\tilde{f}(-k) = \tilde{f}^*(k)$ is a necessary and sufficient condition for $f(x)$ to be real.

(ii) Show that $\tilde{f}(-k) = -\tilde{f}^*(k)$ is a necessary and sufficient condition for $f(x)$ to be pure imaginary.

[4+4]

7. **Fourier transform of an exponential function:** (i) Calculate the Fourier transform of

$$f(t) = \mathrm{e}^{-a|t|} \qquad (a \geq 0).$$

(ii) Calculate the Fourier transform of

$$g(t) = \begin{cases} \mathrm{e}^{-at} & \text{for} \quad t > 0 \\ -\mathrm{e}^{at} & \text{for} \quad t < 0, \end{cases}$$

where $a \geq 0$ as before.

(iii) Using the above results, show that

$$\int_0^{\infty} \mathrm{d}\omega \; \frac{\cos\omega x}{\omega^2 + a^2} = \frac{\pi}{2a} \; \mathrm{e}^{-ax}, \qquad \int_0^{\infty} \mathrm{d}\omega \; \frac{\omega \; \sin\omega x}{\omega^2 + a^2} = \frac{\pi}{2} \; \mathrm{e}^{-ax} \qquad (x > 0).$$

(iv) Now consider another function

$$h(t) = \Theta(t) - \Theta(-t) = \begin{cases} 1 & \text{for} \quad t > 0 \\ -1 & \text{for} \quad t < 0. \end{cases}$$

Show that $\lim_{a \to 0} g(t) = h(t)$, where $g(t)$ is defined in (ii). Then calculate the Fourier transform $\tilde{h}(\omega)$.

(v) Show that we can express the step function as

$$\Theta(t) = \frac{1}{2}[h(t) + 1].$$

What is the Fourier transform of $\Theta(t)$?

[4+4+5+1+4]

8. **Some symmetry properties of Fourier transform:** If $\tilde{f}(\omega)$ is the Fourier transform of $f(t)$, then show that
(i) the Fourier transform of $f(at)$ is $\tilde{f}(\omega/a)/a$,
(ii) the Fourier transform of $f(t-t_0)$ is $\mathrm{e}^{\mathrm{i}\omega t_0}\tilde{f}(\omega)$.

[2+2]

9. **Dirac delta function:** (i) Verify that the function

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\phi')} \qquad (0\le\phi,\phi'\le 2\pi)$$

is a Dirac delta function by showing that it satisfies the definition of a Dirac delta function:

$$\int_0^{2\pi}\mathrm{d}\phi\ f(\phi)\left[\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\phi')}\right]=f(\phi').$$

Hint: Represent $f(\phi)$ by an exponential Fourier series.
(ii) Consider the infinite Dirac comb function

$$g(x)=\sum_{n=-\infty}^{\infty}\delta(x-na).$$

Calculate the Fourier transform of $g(x)$.

[5+3]

10. **Linear filter:** A linear system has been supplied a input $f(t)$ of the form

$$f(t)=\begin{cases}0 & \text{for} \quad t<0\\ \mathrm{e}^{-\lambda t} & \text{for} \quad t>0,\end{cases}$$

where λ is a fixed positive constant, and the output is observed to be

$$g(t)=\begin{cases}0 & \text{for} \quad t<0\\ (1-\mathrm{e}^{-\alpha t})\ \mathrm{e}^{-\lambda t} & \text{for} \quad t>0,\end{cases}$$

where α is another fixed positive constant. Find the transfer function $\tilde{G}(\omega)$ and also find the response of the system to the input $f(t)=\delta(t)$.
Hint: You may need to use the following integral:

$$\int_{-\infty}^{\infty}\mathrm{d}\omega\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega+\mathrm{i}a}=-2\pi\mathrm{i}\mathrm{e}^{-at}\ \Theta(t)$$

[10]

11. **Correlations for the linear filter:** Consider the linear filter having an output

$$g(t)=\int\mathrm{d}\tau\ G(\tau)\ f(t-\tau),$$

where $f(t)$ is the input and $G(\tau)$ is the transfer function. Then show that

$$\mathrm{CCF}(\tau)=\int\mathrm{d}\tau'\ G(\tau')\ \mathrm{ACF}(\tau-\tau'),$$

where the correlation functions are defined as

$$\mathrm{ACF}(\tau)=\int\mathrm{d}t\ f(t)\ f(t-\tau),\quad \mathrm{CCF}(\tau)=\int\mathrm{d}t\ g(t)\ f(t-\tau).$$

[5]

12. **Linear quantum oscillator:** A linear quantum oscillator in its ground state has a wave function

$$\psi(x) = a^{-1/2}\pi^{-1/4}\ \mathrm{e}^{-x^2/2a^2}.$$

Show that the corresponding momentum function is

$$\tilde{\psi}(p) = a^{1/2}\pi^{-1/4}\hbar^{-1/2}\ \mathrm{e}^{-a^2p^2/2\hbar^2}.$$

[5]

13. **Fourier transform of integrals:** Show that if $f(x)$ has a Fourier transform $\tilde{f}(k)$, then the Fourier transform of its integral

$$g(x) = \int_{-\infty}^{x} \mathrm{d}y\ f(y)$$

is given by

$$\tilde{g}(k) = -\frac{\mathrm{i}}{k}\tilde{f}(k) + \pi\tilde{f}(0)\delta(k).$$

[10]

14. **Klein-Gordon equation:** Consider the time-dependent inhomogeneous Klein-Gordon equation:

$$\left[\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + m^2\right]\phi(t,\boldsymbol{x}) = \rho(t,\boldsymbol{x}).$$

Using Fourier transforms, show that the Green's function is given by

$$G(t,\boldsymbol{x},t',\boldsymbol{x'}) = \frac{1}{8\pi^2 R}\int_{-\infty}^{\infty} \mathrm{d}\omega\ \mathrm{e}^{-\mathrm{i}\omega(t-t')}\mathrm{e}^{\pm\mathrm{i}\sqrt{\omega^2-m^2}\ R},$$

where $R = |\boldsymbol{x} - \boldsymbol{x'}|$.

[15]

15. **Slits of finite size:** (i) Consider an opaque screen having two one-dimensional slits of width d placed at $\pm a/2$. The slits are illuminated by a normally incident, unit amplitude monochromatic plane wave of wavelength λ. Show that the intensity distribution of radiation at a distance z from the screen is given by

$$I(x,z) = \left(\frac{2d}{\lambda z}\right)^2\ \mathrm{sinc}^2\left(\frac{\pi x d}{\lambda z}\right)\ \cos^2\left(\frac{\pi x a}{\lambda z}\right).$$

You may assume Fraunhoffer approximation to be valid.

(ii) Repeat the above problem with the apertures being of circular shape both having the same diameter d.

[10+15]

16. **Symmetry properties of DFT:** The functions $f(t_k)$ and $\tilde{f}(\omega_l)$ are discrete Fourier transforms of each other:

$$\tilde{f}(\omega_l) = \frac{T}{N}\sum_{k=0}^{N-1} f(t_k)\ \mathrm{e}^{-\mathrm{i}\omega_l t_k}, \qquad f(t_k) = \frac{1}{T}\sum_{l=0}^{N-1}\tilde{f}(\omega_l)\ \mathrm{e}^{\mathrm{i}\omega_l t_k}.$$

Derive the following symmetry relations:

(i) If $f(t_k)$ is real, then

$$\tilde{f}(\omega_l) = \tilde{f}^*\left(\frac{2\pi N}{T} - \omega_l\right).$$

(ii) If $f(t_k)$ is pure imaginary, then

$$\tilde{f}(\omega_l) = -\tilde{f}^*\left(\frac{2\pi N}{T} - \omega_l\right).$$

[3+3]