Methods of Mathematical Physics I: Assignment 1

IUCAA-NCRA Graduate School

August - September 2012

24 August 2012

To be returned in the class on 5 September 2012

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Equation of radiative transfer:** The transfer of energy through electromagnetic radiation is described the equation of radiative transfer. The equation describes the evolution of a quantity called **specific intensity** $I_\nu(t, \boldsymbol{x}, \hat{\boldsymbol{n}})$ and is given by

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} I_\nu = j_\nu - \kappa_\nu \, I_\nu,$$

where $I_\nu(t, \boldsymbol{x}, \hat{\boldsymbol{n}}) \, \mathrm{d}A \, \mathrm{d}t \, \mathrm{d}\nu \, \mathrm{d}\Omega$ is the energy flowing across an area $\mathrm{d}A$ located at $\boldsymbol{x}$ in the time interval $(t, t + \mathrm{d}t)$ in the solid angle $\mathrm{d}\Omega$ about the direction $\hat{\boldsymbol{n}}$ in the frequency interval $(\nu, \nu + \mathrm{d}\nu)$. The quantity j_ν is the **spontaneous emission coefficient** and κ_ν is the **absorption coefficient**.

(i) Solve the equation (i.e., convert into a set of eight ordinary differential equations) using the method of characteristics. What is the physical significance of the characteristic curves and the parameter s?

(ii) Show that the formal solution of the equation can be written as

$$\begin{aligned} \boldsymbol{x}(t) &= c(t - t_0)\hat{\boldsymbol{n}}; \\ I_\nu(t) &= I_\nu(t_0) \exp\left(-c \int_{t_0}^{t} \mathrm{d}t' \; \kappa_\nu(t')\right) + c \int_{t_0}^{t} \mathrm{d}t' \; j_\nu(t') \exp\left(-c \int_{t'}^{t} \mathrm{d}t'' \; \kappa_\nu(t'')\right). \end{aligned}$$

Give a physical interpretation of the solution.

[5+10]

2. **Traffic flow problem:** Consider the idealized flow of traffic along one-dimension (e.g., a one-lane highway). Let $\rho(t, x)$ be the traffic density at a point x at time t and $v(t, x)$ be the velocity of cars at that point.

(i) Show that if the number of cars is conserved (i.e., no side-way exits), then ρ and v obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0.$$

(ii) Assume that the velocity depends on the density through a relation

$$v(\rho) = c\left(1 - \frac{\rho}{\rho_m}\right),$$

where c is the maximum velocity and ρ_m represents the density during a traffic jam (when $v = 0$). Show that the original differential equation, in terms of suitably re-scaled variables, becomes

$$\frac{\partial u}{\partial t} + (1 - 2u)\frac{\partial u}{\partial x} = 0.$$

(iii) Write down the equations for characteristic curves for this equation. If the initial conditions are given by $x(t = 0) = x_0$, $u(t = 0, x) = f(x)$, then show that the solution is given by

$$x = [1 - 2f(x_0)] \, t + x_0; \quad u = f(x_0).$$

Also note that $\mathrm{d}x/\mathrm{d}t \neq v/c$, i.e., the characteristic velocities do *not* represent the traffic velocity. Rather $\mathrm{d}x/\mathrm{d}t$ is the local speed of the "traffic wave".

(iv) Now consider the initial condition on the traffic density

$$f(x) = \begin{cases} 1 & \text{for } x \leq 0 \\ 0 & \text{for } x > 0, \end{cases}$$

which corresponds to traffic standing at a red light which turns into green at $t = 0$. Show that the solution for the problem is given by

$$x(t) = \begin{cases} x_0 - t & \text{for } x_0 \leq 0 \\ x_0 + t & \text{for } x_0 > 0, \end{cases}$$

and

$$u(t,x) = \begin{cases} 1 & \text{for } x \leq -t \\ 0 & \text{for } x > t. \end{cases}$$

Can you interpret the result?

[3+3+4+5]

3. **Riemann invariants:** The equations of gas dynamics can be expressed as three conservation equations representing conservation of mass, momentum and energy respectively:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} &= 0, \\ \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + p)}{\partial x} &= 0, \\ \frac{\partial E}{\partial t} + \frac{\partial[(E+p)v]}{\partial x} &= 0, \end{aligned}$$

where $E = \rho\, e + \rho v^2/2$ is the total energy per unit volume, with e being the internal energy per unit mass for the fluid. We have assumed the flow to be in one dimension.

(i) Show that the equations can be simplified to give

$$\begin{aligned} \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0, \\ \frac{\partial e}{\partial t} + v\frac{\partial e}{\partial x} + \frac{p}{\rho}\frac{\partial v}{\partial x} &= 0. \end{aligned}$$

(ii) The third equation can be written in a simpler form in terms of the entropy s. Using the thermodynamic relation

$$\mathrm{d}e = T\mathrm{d}s + \frac{p}{\rho^2}\mathrm{d}\rho,$$

show that the equation becomes

$$\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x} = 0.$$

Further, assume an equation of state of the form $p = p(\rho, s)$ and thus eliminate the derivative of p from the second equation (i.e., the momentum conservation equation) in terms of two quantities

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s; \quad \sigma = \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho.$$

The quantity c_s is the local sound speed while σ is a parameter related to the thermal expansitivity of the gas.

(iii) Show that the three equations can be written in a compact notation involving matrices

$$\frac{\partial \mathsf{m}}{\partial t} + \mathsf{A}\cdot\frac{\partial \mathsf{m}}{\partial x} = 0,$$

where

$$\mathsf{m} = \begin{pmatrix} \rho \\ v \\ s \end{pmatrix}.$$

In particular, show that

$$\mathsf{A} = \begin{pmatrix} v & \rho & 0 \\ c_s^2/\rho & v & \sigma \\ 0 & 0 & v \end{pmatrix}.$$

(iv) If the matrix A is diagonalizable, it can be written in the form

$$\mathsf{A} = \mathsf{P} \cdot \Lambda \cdot \mathsf{P}^{-1}.$$

Find the matrices P and Λ.

(v) Now apply P^{-1} from the left to the set of equations written in matrix form and write down the three resultant equations in explicit form.

(vi) Show that one of the equations can be solved using the method of characteristics:

$$s = \text{ const along the characteristics } \frac{\mathrm{d}x}{\mathrm{d}t} = v.$$

(vii) For the other two equations, assume the gas to be isentropic, i.e., $s = \text{const}$. Also assume a polytropic equation of state of the form $p \propto \rho^\gamma$. Write down the two equations using these two simplifications. Show that the two equations are nothing but those written in terms of the Riemann invariants introduced in the class.

[6+5+5+10+9+3+12]

4. **Energy for the wave equation:** Consider the wave equation in one-dimension

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

(i) Define the kinetic and potential energies as

$$K = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x \, \left(\frac{\partial u}{\partial t} \right)^2; \quad P = \frac{c^2}{2} \int_{-\infty}^{\infty} \mathrm{d}x \, \left(\frac{\partial u}{\partial x} \right)^2.$$

Show that the total energy is conserved, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}(K + P) = 0.$$

(ii) Using the energy conservation or otherwise, show that the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad u(t=0, x) = u_0(x); \quad \frac{\partial u(t=0, x)}{\partial t} = v_0(x)$$

has a unique solution.

[5+10]

5. **Gaussian temperature profile:** Consider the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

where κ is the conductivity. Let the initial condition be given by a Gaussian

$$T(t=0, x) = T_0 \mathrm{e}^{-x^2/a^2}.$$

(i) Assume that the temperature profile maintains the Gaussian shape but the parameters may change. Then one can consider a solution of the form

$$T(t, x) = F(t) \; \mathrm{e}^{-H(t) x^2},$$

where $F(t), H(t)$ are to be determined. What are the initial conditions satisfied by these two unknown functions?

(ii) Show that the functions should satisfy the differential equations

$$\frac{\mathrm{d}F}{\mathrm{d}t} = -2\kappa FH; \quad \frac{\mathrm{d}H}{\mathrm{d}t} = -4\kappa H^2.$$

(iii) Solve the two ODEs, put the initial conditions and find the temperature profile $T(t,x)$ for all times.

[3+5+7]

6. **Invariance properties of the diffusion equation:** Show that the diffusion equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

is invariant under the following transformations:

(i) Spatial translations: If $u(t,x)$ is a solution of the diffusion equation, then so is the function $u(t, x-x')$ for any fixed x'.

(ii) Differentiation: If u is a solution of the diffusion equation, then so are $\partial u/\partial x, \partial u/\partial t, \partial u/\partial x^2, \ldots$ and so on.

(iii) Linear combinations: If $u_1, u_2, \ldots, u_n$ are solutions of the diffusion equation, then so is $u = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$ for any constants $c_1, c_2, \ldots, c_n$.

(iv) Integration: If $S(t,x)$ is a solution of the diffusion equation, then so is the integral

$$u(t,x) = \int_{-\infty}^{\infty} \mathrm{d}x' \; S(t, x-x') \; g(x'),$$

for any function $g(x')$ (as long as the integral converges).

(v) Dilation (scaling): If $u(t,x)$ is a solution of the diffusion equation, then so is the dilated function $v(t,x) = u(a\,t, \sqrt{a}\,x)$ for any constant $a > 0$.

[2+2+2+5+4]

7. **Solving the general diffusion equation:** Let us consider the system

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}; \quad v(t=0,x) = \Theta(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x < 0. \end{cases}$$

(i) Show that the above system (i.e., both the PDE and the initial condition) is invariant under the scaling $t \to a^2 t$, $x \to at$. Hence argue that the solution can depend only on the ratio $x/\sqrt{t}$, i.e.,

$$v(t,x) = A\;V\left(\frac{x}{\sqrt{t}}\right) \equiv A\;V(\xi); \quad \xi = \frac{x}{\sqrt{t}},$$

where A is a constant.

(ii) Show that V satisfies the ordinary differential equation

$$V'' + \frac{\xi}{2\alpha} V' = 0.$$

Solve the equation and show that

$$v(t,x) = c_1 \int_0^{x/\sqrt{4\alpha t}} \mathrm{d}\eta \; \mathrm{e}^{-\eta^2} + c_2.$$

(iii) Put in the initial conditions to determine c_1 and c_2.

(iv) Using the invariance properties of the diffusion equation, show that

$$u(t,x) = \int_{-\infty}^{\infty} \mathrm{d}x' \; S(t, x-x') \; u_0(x')$$

is a solution to the diffusion equation, where

$$S(t,x) = \frac{\partial v(t,x)}{\partial x},$$

and $u_0(x)$ is an arbitrary function.

(v) Calculate the initial value $u(t=0,x)$. Hence show that $u(t,x)$ is a solution to the general equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}; \quad u(t=0,x) = u_0(x).$$

(vi) Compute the explicit expression for $S(t,x)$ and show that it is the same propagator derived in the class.

[5+7+5+4+5+4]

8. **Green's function for a damped harmonic oscillator:** Consider a damped harmonic oscillator with a driving force $F(t)$ satisfying the ODE

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = F(t); \qquad t > 0,$$

with the boundary conditions $x(0) = \dot{x}(0) = 0$. Note that the damping coefficient γ and the natural frequency ω_0 are independent of t.

(i) Obtain the solution for the corresponding homogeneous differential equation. Distinguish between the underdamped, critically damped and overdamped cases.

(ii) Find the Green's function(s) and the particular solution(s) for the above system.

(iii) Show that the Green's functions derived above reduce to the familiar form for a forced simple harmonic oscillator (as derived in the class) when $\gamma \to 0$.

[7+10+3]

9. **Asymptotic behaviour of Airy functions:** Consider the Airy differential equation (also known as Stokes equation)

$$y''(x) - xy(x) = 0.$$

Use the WKB method to show that the solutions to the equation (known as Airy functions) behave as

$$y(x) \sim \frac{1}{x^{1/4}} \exp\left(-\frac{2}{3}x^{3/2}\right),$$

when $x \to +\infty$.

[5]

10. **Series solutions of the Laguerre equation:** Consider the Laguerre equation

$$xy'' + (1-x)y' + ky = 0.$$

(i) Find the series solution of the equation by Frobenius' method (show explicitly the first four non-zero terms in the series). Are you able to find both the solutions? For what values of x does the series solution(s) converge? For what values of k does the series become a polynomial?

(ii) Find the second solution of the Laguerre's equation for $k = 0$.

[11+4]

11. **Orthogonality condition for Laguerre polynomials:** Find the orthogonality condition for the solutions of the Laguerre equation:

$$xy'' + (1-x)y' + ky = 0$$

by reducing it to a Sturm Liouville system.

[5]