Mathematical Physics I: Assignment 4

HRI Graduate School

August - December 2010

8 November 2010

To be returned in the class on 18 November 2010

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. Verify the analyticity of the following functions using the Cauchy-Riemann conditions:
(i) $f(z) = z\mathrm{e}^{-z}$
(ii) $f(z) = \mathrm{e}^{z^2}$
(iii) $f(z) = \sinh z$

[$2 \times 3 = 6$]

2. Prove that the following functions $u(x, y)$ are harmonic and find a function $v(x, y)$ such that $u + \mathrm{i}v$ is analytic. Express $u + \mathrm{i}v$ as a function of z.
(i) $u = 2x(1 - y)$
(ii) $u = \mathrm{e}^{-2xy} \sin(x^2 - y^2)$
(iii) $u = \ln[(x - 1)^2 + (y - 2)^2]$

[$4 \times 3 = 12$]

3. If u and v are harmonic in a region, then prove that
$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \mathrm{i}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$
is analytic in the region.

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4. Expand the following functions in a Laurent series about $z = 0$, naming the type of singularity in each case:
(i)
$$\frac{\mathrm{e}^{z^2}}{z^3}$$
(ii)
$$\frac{\cosh(1/z)}{z}$$
(iii)
$$z \sinh \sqrt{z}$$

[$3 \times 3 = 6$]

5. Evaluate the following integrals:
(i)
$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2t\cos\theta + t^2}; \quad |t| < 1$$
(ii)
$$\int_0^{\pi} \mathrm{d}\theta\ \cos^6\theta$$
(iii)
$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta + c\sin\theta}; \quad a^2 > b^2 + c^2$$

[$4 \times 3 = 12$]

6. Evaluate the following integrals (if you are using a large circular-like contour, make sure you show explicitly that the contribution is zero):

(i)
$$\int_0^\infty \mathrm{d}x \, \frac{\sin^2 x}{x^2}$$
(ii)
$$\int_0^\infty \mathrm{d}x \, \frac{\sin ax}{\mathrm{e}^{2\pi x} - 1}$$
(iii)
$$\int_0^1 \frac{\mathrm{d}x}{(x^3 - x^2)^{1/3}}$$
(iv)
$$\int_0^\infty \mathrm{d}x \, \frac{\ln x}{x^2 + a^2}$$

[$4 \times 4 = 16$]

7. The **Legendre polynomials** $P_n(t), n = 0, 1, 2, \ldots$ are defined by **Rodrigues' formula**
$$P_n(t) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} (t^2 - 1)^n$$

(i) Prove that if C is any simple closed curve enclosing the point $z = t$, then
$$P_n(t) = \frac{1}{2\pi \mathrm{i}} \frac{1}{2^n} \oint_C \mathrm{d}z \, \frac{\left(z^2 - 1\right)^n}{(z - t)^{n+1}}$$

This is called **Schlaefli's representation** for $P_n(t)$.

(ii) Prove that
$$P_n(t) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \left(t + \sqrt{t^2 - 1} \, \cos\theta\right)^n$$

[$2 + 3 = 5$]

8. Let
$$G(\sigma) = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\mathrm{e}^{-\mathrm{i}x}}{x^2 - \sigma^2}$$

(i) Compute the principal value of $G(\sigma)$.

(ii) Compute the functions
$$G_\pm(\sigma) = \lim_{\epsilon \to 0^+} G(\sigma \pm \mathrm{i}\epsilon).$$

Make sure you show explicitly that the integrals over the large semi-circles go to zero.

[$6 + 5 = 11$]