Mathematical Physics I: Assignment 3

HRI Graduate School

August - December 2010

30 September 2010

To be returned in the class on 26 October 2010

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. Let V be the vector space of polynomials $\alpha(t)$ having degree ≤ 2 with the inner product $\langle\alpha, \beta\rangle = \int_0^\infty \mathrm{d}t\ w(t)\alpha(t)\beta(t)$ where $w(t) = \mathrm{e}^{-t}$. Start with the usual basis $\{1, t, t^2\}$ and apply Gram-Schmidt algorithm to obtain an orthogonal basis $\{f_0, f_1, f_2\}$. Normalise the new basis such that $f_i(t = 0) = 1$. Can you identify the series of polynomials?

[3]

2. Show that for any pair of vectors α and β in a (complex) inner product vector space

$$||\alpha + \beta|| \leq ||\alpha|| + ||\beta||$$

This is called the **Triangle inequality**.

[2]

3. (i) Consider a linear operator g on the real Euclidean inner product space such that $g(x, y) = (-y, x)$. Show that although $g \neq 0$, we have $\langle g(\alpha)|\alpha\rangle = 0$ for every $\alpha \in V$.
(ii) Show that in a complex inner product space $\langle g(\alpha)|\alpha\rangle = 0$ implies $g = 0$ for any linear operator g.

[$1 + 3 = 4$]

4. Let f be a linear operator on V, and let W be a "f-invariant" subspace of V. Show that $W^\perp$ is invariant under $f^\dagger$.

[2]

5. Let $V = \mathbb{C}^4$ with the standard inner product. Let

$$W = \{x = (x_1, x_2, x_3, x_4) \in V|\sqrt{2}x_1 - x_3 = 0, x_1 - \mathrm{i}x_2 + x_4 = 0\}.$$

(i) Find an orthonormal basis for W.
(ii) Find an orthonormal basis for $W^\perp$.

[$2 + 4 = 6$]

6. Consider the $(1 + 1)$ Minkowski space with coordinates (t, x).
Let $\{e_t = (t = 1, x = 0), e_x = (t = 0, x = 1)\}$ denote the usual basis. Define the coordinates (u, v) through the transformation $u = t - x, v = t + x$.
(i) Define e_u to be the vector $(u = 1, v = 0)$ and $e_v = (u = 0, v = 1)$. Express e_u and e_v in terms of $\{e_t, e_x\}$.
(ii) Show that $\{e_u, e_v\}$ form a basis for vectors in Minkowski space.
(iii) Find the components of the metric tensor g_{ij} on this basis and thus show that e_u and e_v are null and not orthogonal. (They are called a **null basis** for the $t - x$ plane.)
(iv) Let $\{\beta^t, \beta^x\}$ be the basis dual to $\{e_t, e_x\}$. Express the basis $\{\beta^u, \beta^v\}$, which is dual to $\{e_u, e_v\}$, in terms of $\{\beta^t, \beta^x\}$.

[$1 + 1 + 2 + 3 = 7$]

7. Suppose that $\{\alpha_1, \ldots, \alpha_n\}$ is an orthonormal basis for an inner product space V. Let $f : V \to V$ be a linear operator. Define a function $g : V \to V$ by

$$g(\xi) = \sum_{j=1}^{n} \langle f(\alpha_j) | \xi \rangle \, \alpha_j, \quad \xi \in V$$

(i) Prove that g is linear.
(ii) Prove that $g = f^\dagger$.

$[1 + 2 = 3]$

8. Let $V = \mathbb{C}^4$ with the standard inner product. Let f be the linear operator on V whose matrix with respect to the standard basis for V is given by:

$$\mathsf{f} = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

(i) Show that f is a normal operator.
(ii) Find an orthonormal basis for V consisting of eigenvectors of f.
(iii) Find an orthogonal matrix P and a diagonal matrix D such that $\mathsf{D} = \mathsf{P}^{-1}\mathsf{f}\mathsf{P}$.
(iv) Write the spectral decomposition of D in the form $\sum_i \lambda_i E_i$ where E_i are the orthogonal projection matrices.
(v) The above decomposition is in a basis which consists of eigenvectors of f. Transform the spectral decomposition back to the original basis and write f as $\sum_i \lambda_i E_i$ where E_i are the orthogonal projection matrices in the original basis.

$[1 + 3 + 2 + 1 + 3 = 10]$

9. In Euclidean space in Cartesian coordinates, we do not normally distinguish between vectors and one-forms, because their components transform identically. Prove this in two steps:
(i) If the components of a vector transform as

$$\xi'^i = A^i_j \xi^j$$

show the transformation law for a one-form is

$$\psi_i = A'^j_i \psi_j$$

where $\mathsf{A}' = \mathsf{A}^{-1}$. Hence show that the two transformation laws are same if the matrix A is orthogonal.

(ii) The metric of such a Cartesian space has components $g_{ij} = \delta_{ij}$. Prove that a transformation from one Cartesian coordinate system to another must obey

$$\delta_{ij} = A^k_i \delta_{kl} A^l_j$$

and that this implies A is an orthogonal matrix.

$[2 + 1 = 3]$

10. A basis $\{\mathbf{e}_{q^i}\}$ is called a **coordinate basis** if there exists curvilinear coordinates q^i such that

$$\mathbf{e}_{q^i} = \frac{\partial x^j}{\partial q^i} \mathbf{e}_j$$

where x^j are the Cartesian coordinates and $\{\mathbf{e}_j\}$ are the corresponding orthonormal basis.

Consider the polar coordinates (r, θ) in two dimensions.

(i) Calculate the coordinate basis $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ (using the above equation) in terms of the Cartesian basis $\{\mathbf{i}, \mathbf{j}\}$.

(ii) Calculate the orthonormal basis $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta\}$ in terms of $\{\mathbf{i}, \mathbf{j}\}$.

(iii) Assume that the basis $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta\}$ is a coordinate basis, i.e., there exists a coordinate system (ξ, η) such that

$$\hat{\mathbf{e}}_r \equiv \mathbf{e}_\xi = \frac{\partial x}{\partial \xi}\mathbf{i} + \frac{\partial y}{\partial \xi}\mathbf{j}; \qquad \hat{\mathbf{e}}_\theta \equiv \mathbf{e}_\eta = \frac{\partial x}{\partial \eta}\mathbf{i} + \frac{\partial y}{\partial \eta}\mathbf{j}$$

Then compute the quantities

$$\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}$$

Show that

$$\frac{\partial^2 \eta}{\partial y \partial x} \neq \frac{\partial^2 \eta}{\partial x \partial y}$$

What do you conclude from this?

$[2 + 2 + 5 = 9]$

11. Derive the form of the Laplacian of a scalar field in a general coordinate system. What is its form in orthogonal coordinate systems?

[4]

12. (i) Calculate all the Christoffel symbols of second kind for the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}x^2$$

where $a(t)$ is an arbitrary function of t.

(ii) Write the explicit form of the gradient of a scalar field $(\boldsymbol{\nabla} F)^i = g^{ik}\partial F/\partial x^k$, the divergence of a vector field $\boldsymbol{\nabla}\cdot\mathbf{V} = V^i_{;i}$ and the Laplacian $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} F$ in this coordinate system.

(iii) If $a = 1$, the above metric reduces to a simple Minkowskian one. Write the gradient, divergence and Laplacian for the Minkowski metric.

[$6 + 6 + 2 = 14$]

13. A group of N particles is seen to occupy a volume of $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}p_x\,\mathrm{d}p_y\,\mathrm{d}p_z$ in the phase space, so that the number density of particles Π in the phase space is given by:

$$N = \Pi\,\mathrm{d}V\,.$$

Show that Π is invariant under Lorentz transformations.

[3]