Mathematical Physics I: Assignment 2

HRI Graduate School
August - December 2010

13 September 2010
To be returned in the class on 23 September 2010

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. Let V, U, and W are vector spaces over the same field F, and that $f : V \to U$ and $g : V \to W$ are linear mappings. The **composition function** $g \circ f$ is the mapping from V into W defined by $(g \circ f)(\alpha) = g(f(\alpha))$, where $\alpha \in V$. Show that $g \circ f$ is linear whenever f and g are linear.

   [2]

2. Let E be a linear operator on V such that $E^2 = E$. Such an operator is termed a projection. Let U be the image of V and W be the kernel. Show that

   (i) if $\alpha \in U$, then $E(\alpha) = \alpha$, i.e., E is the identity map on U;

   (ii) if $E \neq 1$, then $E(\beta) = 0$ for some $\beta \neq 0 \in V$ (this is called a singular operator), i.e., the kernel is not simply the set $\{0\}$.

   (iii) Calculate $U \cap W$.

   $[2 + 2 + 1 = 5]$

3. Let $M^{22}(\mathbb{R})$ be the vector space of 2×2 matrices over $\mathbb{R}$ and let

   $$\mathsf{X} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

   Let $f : M^{22}(\mathbb{R}) \to M^{22}(\mathbb{R})$ be the mapping defined by $f(\mathsf{A}) = \mathsf{AX} - \mathsf{XA}$. Find a basis and the dimension of the kernel of f. Do the same for the image of f.

   [3]

4. Suppose W is an invariant subspace of $f : V \to V$. Then f has a block matrix representation

   $$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ 0 & \mathsf{C} \end{pmatrix}$$

   where A is a matrix representation of the restriction $\hat{f}$ of f to W.

   [4]

5. Let $f : V \to U$ be an arbitrary linear mapping from a vector space V into a vector space U. Now, for any linear functional $\varphi \in U^*$, the composition $\varphi \circ f$ is a linear mapping from V into F, i.e., $\varphi \circ f \in V^*$. Thus the correspondence $\varphi \to \varphi \circ f$ is a mapping from U^* into V^*; we denote it by f' and call it the **transpose** of f. In other words, $f' : U^* \to V^*$ is defined by

   $$f'(\varphi) = \varphi \circ f.$$

   Thus

   $$(f'(\varphi))(\alpha) = \varphi(f(\alpha))$$

   for every $\alpha \in V, \varphi \in U^*$. Show that the transpose mapping f' defined above is linear.

   [2]

6. Let C be the vector space consisting of the set of real-valued functions $f(x)$ defined on the interval $x \in [-\pi, \pi]$. Let W be the space consisting of functions of the form $g(x) = a \sin x + b \cos x$.

   (i) Show that W is a vector subspace of C.

   (ii) Determine the dimension of W.

$[2 + 1 = 3]$

7. If f is an invertible linear operator on V it is called a **automorphism of** V. It may be thought of as a vector space isomorphism of V onto itself.

   (i) Show that the automorphisms of V form a group with respect to the product law of composition. This is called the **general linear group** on V and denoted $GL(V)$.

   (ii) Show that, unlike $L(V, V)$, $GL(V)$ is not a vector space.

$[3 + 3 = 6]$

8. The **commutator** $[f, g]$ of two operators in $L(V, V)$ is defined as $[f, g] = fg - gh$. Prove the **Jacobi identity** $[[f, g], h] + [[g, h], f] + [[h, f], g] = 0$.

[2]

9. Let f be a linear operator on V. One can define the exponential of a linear operator through the convergent infinite series

$$\mathrm{e}^f \equiv \exp(f) = \sum_{j=0}^{\infty} \frac{f^j}{j!}$$

   Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x, y) = (-y, x)$. Show that

$$\mathrm{e}^{\alpha f} = f \sin \alpha + 1 \cos \alpha$$

   where α is a scalar.

   What is the result of applying $\mathrm{e}^{\alpha f}$ in (x, y)? Can you identify it with something familiar?

[3]

10. (i) Let $\mathbf{v} \in \mathbb{R}^3$ be a nonzero vector. Show that $\Pi_v : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\Pi_v(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$ is a homomorphism of vector spaces over $\mathbb{R}$, where $\mathbf{v} \times \mathbf{w}$ is the vector (cross) product of $\mathbf{v}$ and $\mathbf{w}$. Determine the kernel and the image of Π_v.
    (ii) Let $0 < \theta \leq 2\pi$ be an angle. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the map that associates to a vector v the vector $R_\theta(v)$ obtained performing a (counterclockwise) rotation centered at the origin of angle. Show that R_θ is a homomorphism of vector spaces over $\mathbb{R}$. Determine kernel and image of R_θ.

$[3 + 3 = 6]$

11. Let $f : P_2(\mathbb{C}) \to M^{2,2}(\mathbb{C})$ be the linear transformation defined by

$$f(\alpha(x)) = \begin{pmatrix} 3\alpha(-\mathrm{i}) & \mathrm{i}\alpha(0) \\ \alpha(\mathrm{i}) & \alpha'(0) \end{pmatrix}, \quad \alpha(x) \in P_2(\mathbb{C})$$

   where $P_2(\mathbb{C})$ is the set of polynomials over $\mathbb{C}$ with degree ≤ 2.

   Let $\beta = \{1, x - 1, x^2 - \mathrm{i}x\}$ and $\gamma = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Show that β and γ form bases of $P_2(\mathbb{C})$ and $M^{2,2}(\mathbb{C})$ respectively. Compute the matrix representation of f with respect to the ordered bases β and γ.

[4]

12. Let $f \in L(P_3(\mathbb{R}), P_3(\mathbb{R}))$ be defined by

$$f(ax^3 + bx^2 + cx + d) = ax^3 + (a - 3b + c + d)x^2 + (-2a - b - c - 2d)x + d, \quad a, b, c, d \in \mathbb{R}.$$

   (i) Find the characteristic polynomial and all eigenvalues of f.
   (ii) Determine whether f is diagonalizable. Justify your answer.
   (iii) Find a basis for the eigenspace E_1 of f corresponding to the eigenvalue 1.

$[3 + 3 + 2 = 8]$

13. Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear operator whose effect on a basis $\{e_1, e_2, e_3, e_4\}$ is

$$\begin{aligned} f(e_1) &= 2e_1 - e_4 \\ f(e_1) &= -2e_1 + e_4 \\ f(e_1) &= -2e_1 + e_4 \\ f(e_1) &= e_1 \end{aligned}$$

Find a basis for Ker f and Im f and calculate the rank and nullity of f.

[4]

14. Let $P_4(\mathbb{C})$ be the vector space of polynomials having degree ≤ 4 over a variable t. Let $D : P_4(\mathbb{C}) \to P_4(\mathbb{C})$ and $X : P_3(\mathbb{C}) \to P_4(\mathbb{C})$ be the derivative and multiplication-by-t operators, respectively. Choose $\{1, t, t^2, t^3\}$ as the basis of $P_3(\mathbb{C})$ and $\{1, t, t^2, t^3, t^4\}$ as the basis of $P_4(\mathbb{C})$.
(i) Find the matrix representations of D and X.
(ii) Use the matrix of D so obtained to find the first, second, third, fourth and fifth derivatives of a general polynomial of degree 4.

$[3 + 5 = 8]$