Mathematical Physics I: Assignment 1

HRI Graduate School

August - December 2010

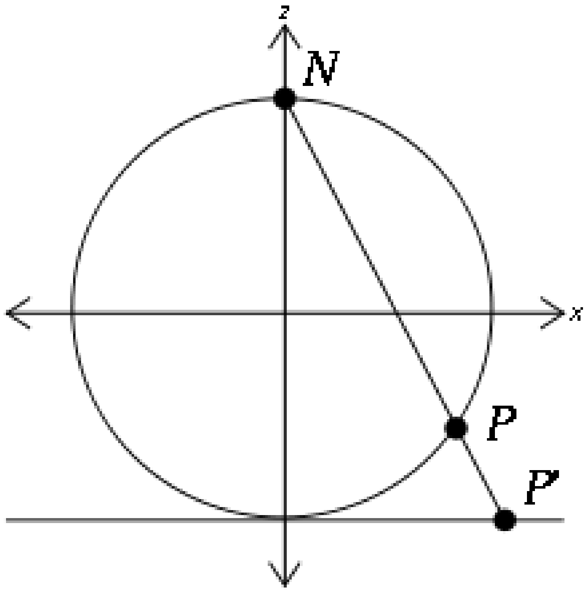
16 August 2010

To be returned in the class on 26 August 2010

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

---

1. The image of a point P on the circle is P', such that P' is the point where the straight line connecting the North Pole N and P intersects the x-axis. Is the map one-to-one?



[1]

2. A **relation** between sets A and B is a subset $\mathcal{R}$ of $A \times B$. If the ordered pair (a, b) lies in $\mathcal{R}$, we read $(a, b) \in \mathcal{R}$ as "a is related to b" and write $a\mathcal{R}b$. For example, let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$ and let $\mathcal{R} = \{(1, y), (1, z), (3, y)\}$. Then $\mathcal{R}$ is a relation from A to B since $\mathcal{R}$ is a subset of $A \times B$. With respect to this relation $1\mathcal{R}y, 1\mathcal{R}z, 3\mathcal{R}y$ but $1\not\mathcal{R}x, 2\not\mathcal{R}x, 2\not\mathcal{R}y, 2\not\mathcal{R}z, 3\not\mathcal{R}x, 3\not\mathcal{R}z$.

   A closed string winds around a cylinder w times. The centre-of-mass momentum along the compact direction of the circle (of radius R) should obey the quantization rule $p = n/R$, where $n \in \mathbb{Z}$. Consider the relation $n\,\mathcal{R}\,w$ defined by

$$\frac{n^2}{R^2} + \frac{w^2\,R^2}{\alpha'^2} = m^2,$$

   where α' and m^2 are constants. Under what conditon is the relation symmmetric?

   [Fun fact: The answer is related to the duality symmetry associated with a compact direction.]

[1]

3. An **equivalence relation** $\mathcal{R}$ on a set S is a special kind of relation, satisfying
   (a) Reflexivity: $x\mathcal{R}x$ for all $x \in S$
   (b) Symmetry: If $x\mathcal{R}y$ then $y\mathcal{R}x$
   (c) Transitivity: If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$

Suppose $\mathcal{R}$ is an equivalence relation on a set S. For each a in S, let $[a]$ denote the set of elements of S to which a is related under $\mathcal{R}$

$$[a] = \{x \mid (a, x) \in \mathcal{R}\}$$

We call $[a]$ the **equivalence class** of a in S under $\mathcal{R}$.

Let $S = \{1, 2, 3, ..., 18, 19\}$. Let R be the relation on S defined by "xy is a square" where $x, y \in S$.
(i) Prove R is an equivalence relation.
(ii) Find the equivalence class [1].
(iii) Examine each element in the set and list all equivalence classes with more than one element.

$[2 + 1 + 3 = 6]$

4. Find the equivalence class of the vector field $A_i(\vec{x})$ such that $\epsilon_{ijk}\partial_j A_k =$ constant.

[1]

5. Prove that if $*$ is an associative and commutative binary operation on a set S, then $(a*b)*(c*d) = [(d*c)*a]*b$ for all $a, b, c, d \in S$. Assume the associative law only for triples as in the definition, that is, assume only $(x*y)*z = x*(y*z)$ for all $x, y, z \in S$.

[2]

6. Either prove the statement, or give a counterexample:
If $*$ and $*'$ are any two binary operations on a set S, then $a * (b *' c) = (a * b) *' (a * c)$ for all $a, b, c \in S$.

[2]

7. Prove the following results for a group $(G, *)$.
(i) The identity element e is unique.
(ii) Each $a \in G$ has a unique inverse a^{-1}.
(iii) $(a^{-1})^{-1} = a$, $(a * b)^{-1} = (b^{-1}) * (a^{-1})$, and, more generally, $(a_1 * a_2 * \ldots * a_n)^{-1} = a_n^{-1} * \ldots * a_2^{-1} * a_1^{-1}$.
(iv) $a * b = a * c$ implies $b = c$, and $b * a = c * a$ implies $b = c$.
(v) G is abelian if and only if $(a * b)^2 = a^2 * b^2$ for all $a, b \in G$.

$[2 + 2 + 4 + 2 + 5 = 15]$

8. Show that every element of $SU(2)$ has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a = d^*$ and $b = -c^*$. Remember that $SU(2)$ is the group of 2×2 unitary matrices with determinant 1 and * denotes complex conjugation.

[3]

9. Consider the ring $\mathbb{Z}_{10} = \{[0], [1], [2], \ldots, [9]\}$ of integers modulo 10.
(i) Find the units of $\mathbb{Z}_{10}$.
(ii) Find the multiplicative inverses of all the units.
(iii) Let $f(x) = 2x^2 + 4x + 4$. Find the roots of $f(x)$ over $\mathbb{Z}_{10}$.

$[2 + 1 + 4 = 7]$

10. Show that $a^2 - b^2 = (a + b) \cdot (a - b)$ for all a and b in a ring $(R, +, \cdot)$ if and only if R is commutative.

[3]

11. Consider $(S, +, \cdot)$, where S is a set and $+$ and $\cdot$ are binary operations on S such that
(a) $(S, +)$ is a group,
(b) $(S^*, \cdot)$ is a group where S^* consists of all elements of S except the additive identity element, and
(c) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ for all $a, b, c \in S$.
Show that $(S, +, \cdot)$ is a division ring.

[5]

12. A ring R is called a "Boolean ring" if $a = a^2$ for every $a \in R$. Show that every Boolean ring is commutative.

[4]