ASTRONOMY AND ASTROPHYSICS: Assignment 2

FERGUSSON COLLEGE, PUNE
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To be uploaded at `https://forms.gle/anUBjiZivmK3rf4X7` on or before 20 May 2020

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not submitted in time.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Orbits in the Schwarzschild spacetime:** Consider a freely moving spaceship in the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - 2GM/r} + r^2\ \mathrm{d}\theta^2 + r^2 \sin^2\theta\ \mathrm{d}\phi^2,$$

where all the symbols have their usual meanings and we are using units where $c = 1$. We have already derived the geodesic equations (assuming $\theta = \pi/2$) for t and ϕ:

$$\left(1 - \frac{2GM}{r}\right)\dot{t} = k,$$
$$r^2\ \dot{\phi} = l,$$

where the overdots represent derivative with respect to the proper time τ.

(a) Using the condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$, find the equation for the radial velocity $\dot{r}$ and radial acceleration $\ddot{r}$. Note that the resulting equations should *not* contain any t-dependent or ϕ-dependent terms.

(b) Calculate the angular speed $\Omega_\infty \equiv \mathrm{d}\phi/\mathrm{d}t$ of the spaceship as a function of r as measured by an observer at infinity.

(c) If we further assume that the spaceship is moving in a circular orbit, we can put $\ddot{r} = \dot{r} = 0$. Using these two conditions, eliminate k and l from the equations and find Ω_∞ as a function of r.

(d) Suppose $r = \alpha\ GM$, where α is a positive number. Then find the period $P_\infty \equiv 2\pi/\Omega_\infty$ of the orbit as a function of α as seen by the observer at infinity.

(e) Compute the angular speed $\Omega_{\mathrm{prop}} \equiv \mathrm{d}\phi/\mathrm{d}\tau$ as measured by an observer in the spaceship in the circular orbit. Also compute the corresponding period P_{prop}.

(f) What happens to P_{prop} when $\alpha < 3$? Can you explain this result?

(g) Find the period of a circular orbit in the Newtonian case and compare with the two periods P_∞ and P_{prop}.

$[4 + 2 + 5 + 2 + 3 + 2 + 2]$