Fluids and Plasmas I: Problem Sheet 4
IUCAA-NCRA Graduate School

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- These problems are for your own practice and will not be graded. They are designed to help you prepare for the mid-term and final examinations. However, I strongly encourage you to ask questions and discuss the solutions.
- If you spot any potential errors or find a question unclear, please do not hesitate to let me know.
- You are welcome to consult books, online resources, and discuss the problems with your peers. The key, however, is to ensure you personally understand the solutions, as this will be vital for your performance in the examinations.
- If you choose to use notation or conventions that differ from those presented in lectures, please define them clearly at the start and apply them consistently.

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1. For astrophysical plasmas such as the solar corona or planetary magnetospheres, currents are often negligible, meaning the magnetic field is approximately a vacuum field ($\vec{\nabla} \times \vec{B} = 0$).

   (a) Prove that for a vacuum magnetic field, the transverse gradient of the field magnitude is strictly related to the radius of curvature vector $\vec{R}_c$ by

   $$\frac{\vec{\nabla}_\perp B}{B} = -\frac{\vec{R}_c}{R_c^2},$$

   where $\vec{\nabla}_\perp$ denotes the component of the gradient perpendicular to the local magnetic field direction. Assume $\vec{R}_c$ points outward from the centre of curvature.

   (b) Using this geometric relation, combine the expressions for gradient drift $\vec{u}_{\nabla B}$ and curvature drift $\vec{u}_c$ to derive the single combined drift velocity formula for a vacuum field.

   (c) Show that for an isotropic velocity distribution, where $\langle u_\parallel^2 \rangle = \langle u_\perp^2 \rangle / 2$, the total drift velocity is proportional to the total kinetic energy $\mathcal{E}$ of the particle.

2. Consider a charged particle trapped in a "magnetic bottle" where the magnetic field strength along the axis (z-direction) varies parabolically

   $$B_z(R, z) = B_0 \left(1 + \frac{z^2}{a^2}\right).$$

   Assume the field to be axisymmetric.

   (a) Show that the radial component of the magnetic field is

   $$B_R(R, z) = -B_0 \frac{zR}{a^2}.$$

   (b) A particle of mass m and charge q is injected at the midplane ($z = 0$) with a total velocity of magnitude u_0, and a pitch angle α_0. Assume adiabatic invariance such that the Larmor radius is much smaller than the characteristic length scale a of the magnetic field variation. Using the conservation of energy and the adiabatic invariant μ, find the z-coordinates of the mirror points, $\pm z_m$, in terms of a and α_0.

(c) Write down the equation of motion for the guiding centre along the field line. Show that the particle undergoes simple harmonic motion along the z-axis.

(d) Derive the "bounce frequency" ω_b of this motion. Does the bounce frequency depend on the particle's total energy?

3. Consider a test charge $+Q$ placed at the origin in an electron-proton plasma.

(a) Write down the exact, non-linear Poisson-Boltzmann equation for the electrostatic potential $\phi(r)$, assuming both electrons and ions follow the Boltzmann distribution $n \propto \mathrm{e}^{\mp e\phi/k_B T}$.

(b) Define a dimensionless potential $\psi = e\phi/k_B T$ and a dimensionless radius $x = r/\lambda_D$, where λ_D is the standard linear Debye length. Show that the exact Poisson equation reduces to the dimensionless form

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}\psi}{\mathrm{d}x}\right) = \sinh(\psi).$$

(c) Show that for $x \gg 1$ (far from the charge), this equation reduces to the linear Debye solution. For $x \ll 1$ (very close to the charge, where $\psi \gg 1$), find an approximate power-law solution for $\psi(x)$ and discuss why complete shielding fails immediately adjacent to a highly charged object.

4. The validity of the Vlasov (collisionless) fluid description relies on the plasma parameter $N_D \equiv n\lambda_D^3 \gg 1$. Calculate the Debye length λ_D (in cm) and the plasma parameter N_D for the following astrophysical environments:

- The warm interstellar medium: $n \sim 1\ \mathrm{cm}^{-3}$, $T \sim 10^4$ K.
- The solar core: $n \sim 10^{26}\ \mathrm{cm}^{-3}$, $T \sim 1.5 \times 10^7$ K.
- A white dwarf core: $n \sim 10^{30}\ \mathrm{cm}^{-3}$, $T \sim 10^7$ K.
- A pulsar magnetosphere: $n \sim 10^{12}\ \mathrm{cm}^{-3}$, $T \sim 10^8$ K.

Based on your calculations, in which of these environments does the ideal plasma approximation ($N_D \gg 1$) break down? For the system(s) that fail the test, what fundamental physical assumption in the derivation of the Maxwell-Boltzmann distribution or Debye shielding is no longer valid?

5. Consider the Vlasov equation for species s, using the microscopic velocity $\vec{u} = \vec{p}/m_s$

$$\frac{\partial f_s}{\partial t} + \vec{u}\cdot\vec{\nabla}_{\vec{x}} f_s + \frac{\vec{F}}{m_s}\cdot\vec{\nabla}_{\vec{u}} f_s = 0,$$

where the Lorentz force is

$$\vec{F} = q_s\left(\vec{E} + \frac{\vec{u}}{c}\times\boldsymbol{B}\right).$$

The momentum equation (first moment) can be derived by multiplying the Vlasov equation by p_a/m_s and integrating. To close the system, an adiabatic equation of state $P_s = K n_s^\gamma$ is assumed. Instead, let us derive the second moment to see where this energy closure originates.

(a) Multiply the Vlasov equation by the kinetic energy term $p^2/2m_s$ and integrate over momentum space d^3p. Define the scalar thermal pressure

$$P_s = \frac{1}{3}m_s\int \mathrm{d}^3w\, f_s w^2,$$

and assume the heat flux vector to be negligible (a valid assumption for an isotropic distribution function)

$$\int \mathrm{d}^3w\, f_s w^2\vec{w} \approx 0,$$

where $\vec{w} = \vec{p}/m_s - \vec{v}_s$ is the random thermal velocity. Show that the second moment yields the energy equation

$$\frac{\partial P_s}{\partial t} + \vec{v}_s\cdot\vec{\nabla}P_s + \frac{5}{3}P_s\left(\vec{\nabla}\cdot\vec{v}_s\right) = 0.$$

(b) Show that this equation reduces exactly to the adiabatic equation of state, and explicitly prove that this recovers $\gamma = 5/3$ for 3D motion.

6. Consider an unmagnetized plasma where both electrons and ions are allowed to move. Assume the electrons are warm with temperature T_e, and the ions are cold ($T_i = 0$).

   (a) Assuming the perturbations to be only along x-direction, write down the linearized one-dimensional continuity and momentum equations for both species, along with Poisson's equation for the electric field.

   (b) Assume plane wave solutions and derive the full two-fluid dispersion relation.

7. The induction equation features a competition between the advection of the magnetic field and its resistive diffusion

$$\frac{\partial \vec{B}}{\partial t} = \nu_m \nabla^2 \vec{B} + \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right),$$

   where the magnetic diffusivity is

$$\nu_m = \frac{c^2}{4\pi\sigma}.$$

   (a) Using the expression for resistivity

$$\eta \approx 7.3 \times 10^{-9}\,\mathrm{s} \left(\frac{T}{\mathrm{K}} \right)^{-3/2} \ln \Lambda,$$

   calculate the electrical conductivity σ and magnetic diffusivity ν_m for the solar corona. Assume $T \approx 2 \times 10^6$ K and a Coulomb logarithm $\ln \Lambda \approx 20$.

   (b) Estimate the characteristic diffusion timescale τ_{diff} for a solar flare loop of characteristic length $L \approx 10^9$ cm.

   (c) Calculate the magnetic Reynolds number $\mathcal{R}_{\text{mag}}$ for this coronal loop, assuming typical fluid velocities of $v \approx 10^7$ cm/s. Based on this number, does the magnetic field diffuse away, or is it frozen into the coronal plasma?

8. Consider a primordial plasma where the initial magnetic field is exactly zero.

   (a) Start with the generalized Ohm's Law (ignoring resistivity and electron inertia, but retaining the electron pressure gradient)

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = -\frac{\vec{\nabla} P_e}{n_e\, e}.$$

   Take the curl of the equation and use Faraday's Law to find an expression for $\partial \vec{B}/\partial t$.

   (b) Assuming the plasma obeys the ideal gas law $P_e = n_e k_B T_e$, show that a magnetic field can only be generated if the density gradients and temperature gradients are not parallel.

   (c) Estimate the magnitude of the magnetic field generated by this "Biermann Battery" mechanism over a timescale of 10^8 years. Assume characteristic length scales for the density and temperature variations are $L \sim 1$ kpc, a temperature of 10^4 K, and standard primordial hydrogen composition.