Fluids and Plasmas I: Problem Sheet 3
IUCAA-NCRA Graduate School

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- These problems are for your own practice and will not be graded. They are designed to help you prepare for the mid-term and final examinations. However, I strongly encourage you to ask questions and discuss the solutions.
- If you spot any potential errors or find a question unclear, please do not hesitate to let me know.
- You are welcome to consult books, online resources, and discuss the problems with your peers. The key, however, is to ensure you personally understand the solutions, as this will be vital for your performance in the examinations.
- If you choose to use notation or conventions that differ from those presented in lectures, please define them clearly at the start and apply them consistently.

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1. While studying surface gravity waves in many astrophysical and geophysical contexts (e.g., shallow accretion disks or ocean waves), the "infinite depth" assumption for the fluid fails. Consider a setup where the upper fluid has $\rho_+ \approx 0$ (air), while the lower fluid has $\rho_- = \rho$ (water). The fluids are static in unperturbed regime $V_\pm = 0$. Both the fluids are incompressible and irrotational.

   Assume the lower fluid has a finite depth h (rigid bottom at $z = -h$) while the upper fluid extends to $z = +\infty$.

   (a) Apply the boundary condition $v_z = 0$ at $z = -h$ to show that the velocity potential in the lower fluid must take the form
   $$\phi \propto \cosh[k(z+h)]\, \mathrm{e}^{\mathrm{i}(kx-\omega t)}.$$

   (b) Derive the new dispersion relation for surface waves
   $$\omega^2 = gk \tanh(kh).$$
   Do you recover the infinite depth result in the limit $kh \gg 1$?

   (c) Show that in the "shallow water" limit ($kh \ll 1$), the waves become non-dispersive with phase velocity $v_p = \sqrt{gh}$.

2. Supernovae remnants and stellar interiors are spheres, not infinite planes. To understand the interface instabilities in such cases, consider a sphere of fluid with density ρ_{in} and radius R, surrounded by an infinite fluid of density ρ_{out}. The system is in equilibrium with a radial gravity $g(R)$ pointing *inward*. Assume both fluids are incompressible and irrotational.

   (a) We perturb the surface radius
   $$r(\theta, \phi, t) = R + \xi, \qquad \xi = A\, Y_l^m(\theta, \phi)\mathrm{e}^{-\mathrm{i}\omega t}.$$
   The perturbed velocity potential φ satisfies $\vec{\nabla}^2\varphi = 0$. In spherical coordinates, show that the solutions that do not diverge are
   $$\varphi_{\text{in}} = B_{\text{in}}\, r^l\, Y_l^m\, \mathrm{e}^{-\mathrm{i}\omega t} \qquad (\text{for } r < R)$$
   $$\varphi_{\text{out}} = B_{\text{out}}\, r^{-(l+1)}\, Y_l^m\, \mathrm{e}^{-\mathrm{i}\omega t} \qquad (\text{for } r > R).$$

(b) Apply the radial velocity matching condition ($\partial\varphi/\partial r = \dot{\xi}$) at $r = R$ to relate constants A, $B_{\rm in}$, and $B_{\rm out}$.

(c) Apply the pressure continuity condition (linearized Bernoulli equation) to derive the dispersion relation for the mode l

$$\omega^2 = -\frac{l(l+1)}{R} g \left[\frac{\rho_{\rm out} - \rho_{\rm in}}{(l+1)\rho_{\rm out} + l\rho_{\rm in}} \right]$$

Verify that for high l (short wavelengths where $l \approx kR$), this reduces to the planar result.

(d) Discuss the stability for the case of $\rho_{\rm in} > \rho_{\rm out}$ versus $\rho_{\rm in} < \rho_{\rm out}$.

3. Consider a viscous, incompressible fluid flowing steadily between two infinite stationary parallel plates located at $y = \pm h$. The flow is driven by a constant pressure gradient $G = -\partial P/\partial x$ in the x-direction.

(a) Starting from the Navier-Stokes equation, show that the velocity profile is parabolic

$$v_x(y) = \frac{G}{2\mu}(h^2 - y^2),$$

where μ is the viscosity coefficient of the fluid.

(b) Compute all components of the viscous stress tensor π_{ij}.

(c) Using the dissipation function

$$\chi = \sum_{ij} \pi_{ij} \frac{\partial v_i}{\partial x_j},$$

calculate the rate of energy dissipation per unit volume as a function of y.

(d) Integrate this over the channel width to find the total power dissipated per unit area of the plates. Show that this matches the work done by the pressure gradient.

4. Consider a *compressible* static fluid with density ρ_0 and pressure P_0. We introduce small perturbations: $\rho = \rho_0 + \rho_1$, $P = P_0 + c_s^2 \rho_1$, and $\vec{v} = \vec{v}_1$, where c_s is the speed of sound, assumed constant. The fluid has a kinematic viscosity coefficient ν which too can be assumed to be constant.

(a) Linearize the compressible Navier-Stokes equation, retaining the viscosity terms. Do not assume $\vec{\nabla} \cdot \vec{v} = 0$.

(b) Take the divergence of the linearized momentum equation and combine it with the linearized continuity equation to derive the equation satisfied by the density perturbation ρ_1.

(c) Assume a plane wave solution $\rho_1 \propto \exp[\mathrm{i}(\vec{k} \cdot \vec{x} - \omega t)]$ and show that the dispersion relation is

$$\omega^2 = k^2 c_s^2 - \mathrm{i}\omega k^2 \left(\frac{4}{3}\nu \right).$$

(d) For small viscosity ($\nu \ll c_s^2/\omega$), solve for ω and find the spatial attenuation length of the sound wave.

5. The equation describing evolution of a thin accretion disk is given by

$$\frac{\partial \Sigma(R,t)}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2} \frac{\partial}{\partial R}\left(\nu\, \Sigma(R,t)\, R^{1/2} \right) \right],$$

where $\Sigma(R,t)$ is the surface density and ν is the coefficient of viscosity.

(a) Assume ν to be a constant. Define dimensionless variables

$$x = \frac{R}{R_0}, \qquad \tau = \frac{12\nu t}{R_0^2},$$

where R_0 is some characteristic scale, to show that the equation becomes

$$\frac{\partial \Sigma(x,\tau)}{\partial \tau} = \frac{1}{4x}\frac{\partial}{\partial x}\left[x^{1/2} \frac{\partial}{\partial x}\left(\Sigma(x,\tau)\, x^{1/2} \right) \right].$$

(b) Define $s = 2x^{1/2}$ and $F = x^{1/2}\,\Sigma$, and show that the equation further reduces to

$$\frac{\partial F(s,\tau)}{\partial \tau} = \frac{1}{s^2}\frac{\partial^2 F(s,\tau)}{\partial s^2}.$$

(c) The equation can be solved numerically using the difference equation

$$F_{i,\alpha+1} = F_{i,\alpha} + \frac{\Delta\tau}{(\Delta s)^2}\frac{1}{s_i^2}\left[F_{i+1,\alpha} - 2F_{i,\alpha} + F_{i-1,\alpha}\right],$$

where the coordinates are discretized as

$$\begin{aligned} s_i &= i\,\Delta s, & i &= 1, 2, \ldots, \\ \tau_\alpha &= \alpha\,\Delta\tau, & \alpha &= 1, 2, \ldots, \end{aligned}$$

and the unknown function is discretized as $F_{i,\alpha} = F(s_i, \tau_\alpha)$. Write a numerical code to find the solution with the initial condition

$$\Sigma(x, \tau = 0) = \exp\left[-\frac{(x-1)^2}{0.005}\right].$$

Plot the solutions $\Sigma(x,\tau)$ for $\tau = 0, 0.05, 0.1, 0.15, 0.2$ in the range $0.01 \le x \le 1.5$. All the curves should be on the same plot. Also mention the values of $\Delta\tau$ and Δs used, preferably in the title of the plot.

Your submission should be file containing the numerical code you have written. The file should also contain instructions on how to (compile and) run the code, preferably as a comment at the beginning of the code. Once the code is run, it should produce the plot.

Hint: Make sure you choose the values of $\Delta\tau$ and Δs appropriately, otherwise the solutions will not be numerically stable.

6. For a turbulent fluid, the velocity correlation tensor is given by

$$\overline{v_i(\vec{x})\,\vec{v}_j(\vec{x}+\vec{r})} = \frac{\overline{v^2}}{3}\left[g(r)\delta_{ij} + (f(r) - g(r))\frac{r_i r_j}{r^2}\right], \qquad g(r) = f(r) + \frac{r}{2}\frac{\mathrm{d}f(r)}{\mathrm{d}r},$$

where $f(r)$ is the longitudinal correlation function.

Show that the energy spectrum $E(k)$ (the kinetic energy per unit mass per unit wavenumber) can be written in terms of the longitudinal correlation function $f(r)$ as

$$E(k) = \frac{\overline{v^2}}{3\pi}\int_0^\infty \mathrm{d}r\, f(r)\left[2kr\sin(kr) - (kr)^2\cos(kr)\right].$$

7. Consider the warm neutral medium (WNM) of the ISM with the following typical parameters:

- Outer scale of turbulence (injection scale): $L \sim 100$ pc
- RMS velocity fluctuations at scale L: $V_L \sim 10$ km s^{-1}
- Number density: $n \sim 1$ cm^{-3}
- Temperature: $T \sim 8000$ K

(a) Estimate the kinematic viscosity ν of the gas. Assume $\nu \approx \frac{1}{3}v_{\rm th}\lambda_{\rm mfp}$, where $v_{\rm th} = \sqrt{k_B T/m_p}$ and the mean free path $\lambda_{\rm mfp} \approx 1/(n\sigma)$ with cross-section $\sigma \approx 10^{-15}$ cm^2 (appropriate for neutral hydrogen collisions). Compare the value with that of water ($\nu \approx 0.01$ cm^2 s^{-1}) and explain the physical reason for the difference.

(b) Calculate the Reynolds number $\mathcal{R}$ for the WNM.

(c) Calculate the Kolmogorov length scale $\eta = (\nu^3/\epsilon)^{1/4}$, where ϵ is the energy dissipation rate per unit mass. Express your answer in Astronomical Units (AU).

(d) Compare η to the mean free path $\lambda_{\rm mfp}$. Is the assumption of a fluid continuum valid at the dissipation scale?

8. In the theory of turbulent diffusion, the displacement of a fluid particle is given by

$$\vec{x}(t) = \int_0^t \mathrm{d}t' \, \vec{v}_L(t'),$$

where $\vec{v}_L(t)$ is the Lagrangian velocity of the particle. The mean square displacement $\overline{\vec{x}^2(t)}$ can be expressed in terms of the Lagrangian velocity correlation function

$$R(\tau) = \frac{\overline{\vec{v}_L(t) \cdot \vec{v}_L(t-\tau)}}{\overline{v^2}},$$

where $\overline{v^2}$ is the mean square velocity.

(a) Show that

$$\overline{\vec{x}^2(t)} = 2\overline{v^2} \int_0^t \mathrm{d}\tau \, (t-\tau) \, R(\tau).$$

(b) Assume the Lagrangian velocity correlation function decays exponentially:

$$R(\tau) = \mathrm{e}^{-|\tau|/\tau_{\mathrm{cor}}},$$

where τ_{cor} is the correlation time.
Perform the integration explicitly to find the expression for $\overline{\vec{x}^2(t)}$ valid for all times t. Show that for $t \ll \tau_{cor}$, the particles move ballistically ($\overline{\vec{x}^2} \propto t^2$). Show that for $t \gg \tau_{\mathrm{cor}}$, the particles move diffusively ($\overline{\vec{x}^2} \propto t$). Define an effective turbulent diffusion coefficient D_{turb} from your result.