Fluids and Plasmas I: Problem Sheet 2
IUCAA-NCRA Graduate School

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- These problems are for your own practice and will not be graded. They are designed to help you prepare for the mid-term and final examinations. However, I strongly encourage you to ask questions and discuss the solutions.
- If you spot any potential errors or find a question unclear, please do not hesitate to let me know.
- You are welcome to consult books, online resources, and discuss the problems with your peers. The key, however, is to ensure you personally understand the solutions, as this will be vital for your performance in the examinations.
- If you choose to use notation or conventions that differ from those presented in lectures, please define them clearly at the start and apply them consistently.

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1. Consider a plane-parallel isothermal atmosphere with constant sound speed c_s and uniform gravity g pointing in the $-z$-direction. Assume the gas to be inviscid. Assume an ideal gas equation of state.

   (a) The unperturbed atmosphere is in hydrostatic equilibrium. Find the dependence of the unperturbed density $\rho = \rho_0(z)$ and show that it can be written as $\rho_0(z) \propto \mathrm{e}^{-z/H}$. Write the scale height H in terms of c_s^2 and g. What is the velocity $\vec{v}$ of the unperturbed system?

   (b) Now assume that the system is perturbed and the perturbations are so small that we can work in the linear order. If ρ_1 and v_{1z} are density and velocity perturbations, respectively, then use the continuity and Euler equations to write the equations satisfied by ρ_1 and v_{1z}. Manipulate these equations to write a single equation for ρ_1.

   (c) Assume solutions of the form $\rho_1(z,t) = \tilde{\rho}_1(z)\,\mathrm{e}^{-\mathrm{i}\omega t}$, and write the differential equation satisfied by $\tilde{\rho}_1(z)$. What are the solutions of this equation?

   (d) What should be the condition on ω for the system have propagating sound waves? Write down the full solution for $\rho_1(z,t)$ in such case.

   (e) Assuming the system allows sound waves, what is the dispersion relation between ω and wave number k? Give a qualitative plot of the dispersion relation ω vs k.

   (f) Show that the velocity perturbations have solutions of the form $v_{1z} \propto \mathrm{e}^{z/2H}\mathrm{e}^{\pm \mathrm{i}kz - \mathrm{i}\omega t}$. What happens to these perturbations with increasing height of the atmosphere?

2. Consider a general compressible fluid element in a star in hydrostatic equilibrium with background density $\rho_0(z)$, pressure $P_0(z)$, and temperature $T_0(z)$, where we have assumed plane stellar geometry with z being the vertical coordinate (pointing upwards). The gravitational acceleration is g downwards (constant). Consider a "blob" displaced adiabatically by ξ upwards, with its pressure equalizing with the surroundings immediately.

   (a) Show that the density difference between the blob and the environment at $z + \xi$ is

$$\delta\rho = \rho_{\text{blob}} - \rho_0 = \left[-\left(1 - \frac{1}{\gamma}\right) \frac{\rho_0(z)}{P_0(z)} \left(\frac{\mathrm{d}P_0}{\mathrm{d}z}\right) + \frac{\rho_0(z)}{T_0(z)} \left(\frac{\mathrm{d}T_0}{\mathrm{d}z}\right) \right] \xi,$$

   where assume the fluid to follow the ideal gas law and γ is the adiabatic index.

(b) Show that the equation of motion for the blob can be written as

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + N^2\xi = 0,$$

where N is the Brunt-Väisälä frequency defined as

$$N^2 = \frac{g}{T_0(z)}\left[\left(\frac{\mathrm{d}T_0}{\mathrm{d}z}\right) - \left(1 - \frac{1}{\gamma}\right)\frac{T_0(z)}{P_0(z)}\left(\frac{\mathrm{d}P_0}{\mathrm{d}z}\right)\right].$$

Hence derive the Schwarzschild Criterion for stability against convection.

3. Real fluids have viscosity, which dissipates energy. Let us model such fluid by adding a kinematic viscosity term to the linearized momentum equation for perturbations

$$\frac{\partial \vec{v}_1}{\partial t} = -\frac{c_s^2}{\rho_0}\nabla\rho_1 + \nu\nabla^2\vec{v}_1.$$

Assume no gravity for this problem.

(a) Assume a plane wave solution propagating along the x-axis $\propto \mathrm{e}^{\mathrm{i}(kx-\omega t)}$ and derive the dispersion relation. Use the linearized continuity equation with uniform background density.

(b) Show that for small viscosity ($\nu \ll c_s/k$), the wave oscillates at the sound speed but decays exponentially with a timescale τ. Find the expression for the decay timescale τ.

4. Consider the Burger's equation in the presence of viscosity

$$\frac{\partial v(x,t)}{\partial t} + v(x,t)\frac{\partial v(x,t)}{\partial x} = \nu\frac{\partial^2 v(x,t)}{\partial x^2},$$

where ν is the coefficient of viscosity, assumed to be constant.

Let us assume that there exist travelling wave solutions of the form

$$v(x,t) = v(x - c_0 t),$$

where $c_0 > 0$ is a constant. The initial conditions are given by

$$\begin{aligned} v(x, t=0) &= v_2 && \text{at } x \to -\infty, \\ &= 0 && \text{at } x \to \infty, \\ &= \frac{v_2}{2} && \text{at } x = 0, \\ \frac{\partial v(x,t=0)}{\partial x} &= 0 && \text{at } x \to \pm\infty, \end{aligned}$$

where $v_2 > 0$.

(a) Define $s = x - c_0 t$ and show that the Burger's equation reduces to a ordinary differential equation

$$\nu\frac{\mathrm{d}^2 v}{\mathrm{d}s^2} - v\frac{\mathrm{d}v}{\mathrm{d}s} + c_0\frac{\mathrm{d}v}{\mathrm{d}s} = 0.$$

(b) Integrate the equation once. Use the boundary conditions to show that the resulting differential equation has the form

$$\nu\frac{\mathrm{d}v}{\mathrm{d}s} - \frac{1}{2}v^2 + c_0 v = 0,$$

and the constant is given by $c_0 = v_2/2$.

(c) Integrate once more and obtain the solution $v(x,t)$, ensuring all boundary conditions are satisfied.

(d) Make qualitative plots of the solutions for different times, and for different values of viscosity ν, and interpret them.

(e) Can you use the solutions to characterize the thickness of the shock?

5. Consider a weak shock where the pressure jump is small, i.e.,

$$\frac{P_2}{P_1} = 1 + \delta, \qquad \delta \ll 1.$$

(a) Using the Rankine-Hugoniot relations, show that the density jump is given by

$$\frac{\rho_2}{\rho_1} = 1 + \frac{\delta}{\gamma} + \frac{1-\gamma}{2\gamma^2}\delta^2 + \frac{(\gamma-1)^2}{4\gamma^3}\delta^3 + \mathcal{O}(\delta^4).$$

Assume an ideal gas with constant adiabatic index γ.

(b) Derive the expression for the entropy change $\Delta s = s_2 - s_1$ across the shock for an ideal gas and show that the entropy jump is of the third order in shock strength

$$\frac{\Delta s}{c_v} \approx \frac{\gamma^2 - 1}{12\gamma^2}\delta^3.$$

Interpret the physical meaning of this result.

6. The Sedov-Taylor solution is usually derived for a uniform medium ($\rho =$ const). However, many supernovae occur in the wind of the progenitor star (e.g., Wolf-Rayet stars), where density decreases with radius.

(a) Let a supernova with energy E explode into a surrounding medium with a density profile $\rho(r) = Ar^{-\alpha}$, where A is a constant and $\alpha < 3$. Using the assumption of self-similarity and dimensional analysis, derive the power-law expansion scaling for the shock radius $r_{\text{sh}}(t)$ in terms of $E, A, t,$ and α.

(b) Show that for the specific case of a steady stellar wind ($\alpha = 2$), the shock expands as $r_{\text{sh}} \propto t^{2/3}$.

(c) Calculate the shock velocity $v_{\text{sh}}(t)$ for the $\alpha = 2$ case. Does the shock decelerate faster or slower than in the uniform medium case ($\alpha = 0$)? Explain the physical reason.

7. Consider a young supernova remnant (SNR) approximately 1000 years after the explosion. Assume it is in the Sedov-Taylor phase. The explosion energy is $E = 10^{51}$ erg and the surrounding ISM number density is $n_0 = 1$ cm^{-3} (assume mean molecular weight $\mu \approx 0.6$).

(a) Calculate the current radius r_{sh} (in parsecs) and the shock velocity v_{sh} (in km/s).

(b) Using the strong shock approximation, calculate the immediate post-shock temperature T_s. Express your answer in Kelvin and in keV ($k_B T$).

(c) If the dominant cooling mechanism is thermal bremsstrahlung, the cooling rate per unit volume is $\Lambda \approx 2 \times 10^{-27} n_e n_i T^{1/2}$ erg cm^{-3} s^{-1}. Estimate the cooling time $t_{\text{cool}} \approx 3nk_B T/\Lambda$ for the post-shock gas. Compare t_{cool} with the age of the SNR. Is the assumption of an adiabatic (Sedov) shock justified?

8. In astrophysical disks and planetary atmospheres, it is often convenient to work in a rotating reference frame. Consider a frame rotating with a constant angular velocity $\vec{\Omega}$ relative to an inertial frame. The Euler equation in this frame includes the Coriolis and centrifugal forces.

(a) Show that the centrifugal term can be written as the gradient of a potential: $\vec{\Omega} \times (\vec{\Omega} \times \vec{x}) = \vec{\nabla}\Phi_{\text{cent}}$. Find the expression for Φ_{cent}.

(b) Show that for a steady barotropic flow in this rotating frame, the quantity

$$H_{\text{rot}} = \frac{1}{2}v^2 + \int \frac{dP}{\rho} + \Psi + \Phi_{\text{cent}}$$

is conserved along streamlines. Does the Coriolis force play a role in the energy budget? Explain why or why not based on your derivation.

9. Consider a spherically symmetric accretion of gas onto a star of mass M_*. Let the flow of gas be steady, i.e., the mass accretion rate is constant, and no physical quantity has any explicit dependence on time. The radial velocity $v_r(r)$ can be taken to be a monotonically decreasing function of r.

   (a) Starting from the continuity and Euler equations, show that $v_r(r_c) = c_s(r_c)$, where the critical radius r_c is defined through the equation
   $$r_c = \frac{GM_*}{2c_s^2(r_c)}.$$
   The quantity $c_s(r)$ is the sound speed which, in general, is a function of r. You can assume that the gravity is provided entirely by the central mass.

   (b) Let us assume an equation of state of the form $P \propto \rho^\gamma$, with $1 < \gamma < 5/3$. Show that
   $$\frac{1}{2}v_r^2 + \frac{c_s^2}{\gamma - 1} - \frac{GM_*}{r} = \text{constant}.$$

   (c) Fix the constant using the boundary condition that $v_r \to 0$ at $r \to \infty$ and show that
   $$\frac{c_s^2}{\gamma - 1} = -\frac{1}{2}v_r^2 + \frac{c_{s,\text{ISM}}^2}{\gamma - 1} + \frac{GM_*}{r},$$
   where $c_{s,\text{ISM}} = c_s(r \to \infty)$.

   (d) It is possible to relate the quantities at $r \to \infty$ to those at $r = r_c$. In particular, show that
   $$c_s^2(r_c) = c_{s,\text{ISM}}^2 \frac{2}{5 - 3\gamma},$$
   $$\rho(r_c) = \rho_{\text{ISM}} \left(\frac{2}{5 - 3\gamma}\right)^{1/(\gamma-1)},$$
   where $\rho_{\text{ISM}} = \rho(r \to \infty)$.

   (e) Show that the accretion rate is given by
   $$\dot{M} = \pi (GM_*)^2 \frac{\rho_{\text{ISM}}}{c_{s,\text{ISM}}^3} \left(\frac{2}{5 - 3\gamma}\right)^{(5-3\gamma)/(2\gamma-2)}.$$
   Write down the expression for $\gamma = 5/3$.

   (f) Using typical values of density and temperature of the interstellar medium (ISM), and assuming $M_* \sim M_\odot$, find an order of magnitude estimate of $\dot{M}$ in units of $M_\odot$/yr. Suppose 1% of this mass in converted into radiation, then estimate the luminosity of the system in units of erg s^{-1}. Compare this luminosity with typical solar luminosity.

10. Consider the relation between angular velocity Ω and eccentricity e for a Maclaurin spheroid
   $$\frac{\Omega^2}{2\pi G\rho_0} = \frac{\sqrt{1 - e^2}}{e^3}(3 - 2e^2)\sin^{-1}(e) - \frac{3(1 - e^2)}{e^2}$$

   (a) Perform a rigorous asymptotic expansion of this expression in the limit $e \to 0$ (slow rotation) to show that $\Omega^2 \propto e^2$.

   (b) Perform the expansion in the limit $e \to 1$ (high flattening). Show that Ω does not behave monotonically but reaches a maximum.

   (c) **Numerical Task:** Plot the function $\Omega^2/(2\pi G\rho_0)$ vs e for $0 \le e < 1$. Determine the value of e (to 3 significant figures) where the rotation speed is maximized. What is Ω_{max} in units of $\sqrt{G\rho_0}$?

11. Consider a star that is significantly flattened due to rapid rotation. Assume it can be modelled as a central point mass M with a massless rotating envelope.

(a) Show that for such a model, the surface of the star is defined by

$$\Phi_{\text{eff}} = -\frac{GM}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta = \text{const.}$$

(b) Using Von Zeipel's theorem ($T_{\text{eff}} \propto g_{\text{eff}}^{1/4}$), where g_{eff} is the local *effective* gravity, derive an expression for the ratio of the effective temperature at the pole (T_p) to the effective temperature at the equator (T_{eq}) in terms of the ratio of polar to equatorial radii (R_p/R_{eq}).

(c) What happens when $R_{\text{eq}} = 1.5R_p$? Explain the result physically.