Fluids and Plasmas I: Problem Sheet 1
IUCAA-NCRA Graduate School

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- These problems are for your own practice and will not be graded. They are designed to help you prepare for the mid-term and final examinations. However, I strongly encourage you to ask questions and discuss the solutions.
- If you spot any potential errors or find a question unclear, please do not hesitate to let me know.
- You are welcome to consult books, online resources, and discuss the problems with your peers. The key, however, is to ensure you personally understand the solutions, as this will be vital for your performance in the examinations.
- If you choose to use notation or conventions that differ from those presented in lectures, please define them clearly at the start and apply them consistently.

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1. Consider a gas of classical particles described by the Boltzmann equation for a distribution function $f(\vec{x}, \vec{p}, t)$. Show that, for a uniform gas with no external forces, the quantity

$$H = \int \mathrm{d}^3p\, f(\vec{p}, t)\ \ln f(\vec{p}, t),$$

can never increase with time. Hence show that the distribution function approaches the Maxwell-Boltzmann distribution at late times. Assume binary elastic collisions and the standard Boltzmann collision integral.
Hint: You may need to use the fact that elastic collisions are invariant under time reversal and parity.

2. Show that the pressure tensor P_{ij} corresponding to an isotropic distribution is diagonal $P_{ij} = P\ \delta_{ij}$. Also show that

$$P = \frac{4\pi}{3m} \int_0^\infty \mathrm{d}p\, p^4\ f(\vec{x}, p, t).$$

Assuming f to be given by a Maxwell-Boltzmann distribution

$$f \propto \exp\left(-\frac{p^2}{2mk_BT}\right),$$

show that $P = nk_BT$.

3. Consider a collection of photons. Let the distribution function be $f(\vec{x}, \vec{p}, t)$.

   (a) Show that the number density of photons n_γ and the energy density u_γ at a position $\vec{x}$ can be expressed as:

$$n_\gamma = \int \mathrm{d}^3p\, f, \quad u_\gamma = \int \mathrm{d}^3p\, (cp)\, f.$$

   (b) In radiative transfer, we use specific intensity $I_\nu(\vec{x}, \hat{n}, t)$, defined as the energy per unit area, per unit time, per unit solid angle, per unit frequency. Using the relation $\mathrm{d}^3p = p^2\, \mathrm{d}p\, \mathrm{d}\Omega$ and $p = h\nu/c$, prove that

$$I_\nu = \frac{h^4\nu^3}{c^2} f.$$

(c) In the absence of interactions (collisions), the distribution function follows the collisionless Boltzmann Equation (CBE)

$$\frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \vec{\nabla} f + \dot{\vec{p}} \cdot \vec{\nabla}_p f = 0.$$

For photons in a vacuum (no gravity), $\dot{\vec{p}} = 0$. Using $\dot{\vec{x}} = c\hat{n}$ (where $\hat{n}$ is the unit vector of propagation), show that the CBE reduces to

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{n} \cdot \vec{\nabla} I_\nu = 0,$$

which is the radiative transfer equation in vacuum.

(d) The differential equation can be solved by the method of characteristics. Show that the "characteristic curves" are straight lines in space-time defined by:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}s} = \hat{n},$$

where s is an affine parameter along the curve. Show that along these specific trajectories (the ray paths), the specific intensity I_ν remains constant, i.e.,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0.$$

4. Consider the internal energy equation for fluids

$$\rho \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -P\left(\vec{\nabla} \cdot \vec{v}\right) - \vec{\nabla} \cdot \vec{q} + \sum_{i,j} \pi_{ij} \frac{\partial v_j}{\partial x_i},$$

where $\mathcal{E}$ is the specific internal energy, P is the pressure, $\vec{v}$ is the fluid velocity, $\vec{q}$ is the heat flux vector, and π_{ij} is the viscous stress tensor.

(a) Using the internal energy equation and assuming the thermodynamic relation for an ideal gas

$$T\mathrm{d}s = \mathrm{d}\mathcal{E} + P\mathrm{d}\left(\frac{1}{\rho}\right),$$

where s is the specific entropy, derive the evolution equation for entropy.

(b) Show that in the absence of heat flux and viscosity (Euler limit), the flow is isentropic (adiabatic).

(c) Using the Fourier law ($\vec{q} = -\mathcal{K}\vec{\nabla}T$) and the Newtonian viscous stress

$$\pi_{ij} = 2\mu\left(\Lambda_{ij} - \frac{1}{3}\delta_{ij}(\vec{\nabla} \cdot \vec{v})\right), \quad \Lambda_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

where μ is the coefficient of shear viscosity, show that the entropy of the system must monotonically increase (integrated over a closed volume), consistent with the Second Law of Thermodynamics.

5. The Chapman-Enskog procedure requires the Knudsen number

$$\mathrm{Kn} = \frac{\lambda}{L} \ll 1,$$

where λ is the mean free path of particles in the fluid, and L is the characteristic macroscopic length scale over which fluid properties vary significantly. In astrophysics, "fluids" are often extremely rarefied, and this assumption must be checked.

Consider two astrophysical systems:

(a) The Solar Corona: $n \approx 10^9$ cm^{-3}, $T \approx 10^6$ K, length scale $L \approx R_\odot \approx 7 \times 10^{10}$ cm.

(b) The core of the Intra-Cluster Medium (ICM) of a Galaxy Cluster: $n \approx 10^{-3}$ cm^{-3}, $T \approx 10^8$ K, length scale $L \approx 100$ kpc.

Assume the gas is fully ionized hydrogen in both cases. The effective Coulomb collision cross-section can be approximated as (in Gaussian units):

$$\sigma \approx \frac{\pi e^4}{(k_B T)^2} \ln \Lambda,$$

where the Coulomb logarithm $\ln \Lambda \approx 20$.

Calculate the mean free path $\lambda = (n\sigma)^{-1}$ for both systems, and compute the Knudsen number for both. Discuss if the standard fluid approximation valid for describing global flows in these systems.

6. Start with the Eulerian equation of motion for a self-gravitating fluid:

$$\rho \frac{\partial v_j}{\partial t} + \rho \sum_k v_k \frac{\partial v_j}{\partial x_k} = -\sum_k \frac{\partial P_{jk}}{\partial x_k} - \rho \frac{\partial \Phi}{\partial x_j},$$

where Φ is the gravitational potential and P_{jk} is the pressure tensor. Multiply this equation by x_i and integrate over the entire volume V of the system (assume vacuum boundary conditions, i.e., $\rho = 0$ and $P_{jk} = 0$ at the surface). Hence show that the tensor Virial Theorem holds:

$$\frac{1}{2}\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = 2\mathcal{T}_{ij} + \Pi_{ij} + W_{ij}$$

where:

- $I_{ij} = \int_V \mathrm{d}^3x \, \rho x_i x_j$ is the moment of inertia tensor,
- $\mathcal{T}_{ij} = \frac{1}{2}\int_V \mathrm{d}^3x \, \rho v_i v_j$ is the kinetic energy tensor,
- $\Pi_{ij} = \int_V \mathrm{d}^3x \, P_{ij}$ is the pressure integral,
- $W_{ij} = -\int_V \mathrm{d}^3x \, \rho x_i \frac{\partial \Phi}{\partial x_j}$ is the potential energy tensor.

Recover the standard scalar Virial Theorem for a static self-gravitating system.

7. Consider two fluid elements located at $\vec{x}$ and $\vec{x} + \delta\vec{l}$ at time t. As the fluid flows, both elements move, and the vector $\delta\vec{l}$ connecting them evolves.

   (a) Using the definition of the material (Lagrangian) derivative $\mathrm{d}/\mathrm{d}t$, show that the rate of change of this line element is given by:

   $$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\vec{l}) = (\delta\vec{l} \cdot \vec{\nabla})\vec{v}.$$

   (b) Write the above equation in the component form

   $$\frac{\mathrm{d}}{\mathrm{d}t}(\delta l_i) = \sum_j H_{ij} \delta l_j,$$

   where H_{ij} is the velocity gradient tensor that completely determines how a fluid element changes its shape and volume. Show that H_{ij} can be decomposed into three physically distinct parts:

   $$H_{ij} = \frac{1}{3}\theta\delta_{ij} + \sigma_{ij} + \omega_{ij},$$

   where

   $$\theta = \vec{\nabla} \cdot \vec{v}$$

   is the expansion scalar,

   $$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3}\theta\delta_{ij}$$

is the shear tensor (trace-free symmetric) and

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

is the vorticity tensor (antisymmetric).

(c) Consider a small fluid element that is spherical at time $t = 0$. Describe how the element looks at $t + \delta t$ due to the action of only one of these terms at a time. What does θ do to the sphere? What does σ_{ij} do to the sphere? What does ω_{ij} do to the sphere?

(d) Taking the divergence of the Euler equation for a self-gravitating fluid:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\Phi,$$

and using the decomposition of H_{ij}, derive the evolution equation for the expansion scalar θ (known as the Newtonian analogue of Raychaudhuri equation used in General Relativity):

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 - 4\pi G\rho + \text{pressure terms}$$

where $2\sigma^2 \equiv \sigma_{ij}\sigma_{ji}$ and $2\omega^2 \equiv \omega_{ij}\omega_{ji}$.

(e) Based on the equation derived for θ, discuss the role of shear (σ) and vorticity (ω) in the context of gravitational collapse. Does shear assist or resist the collapse of a gas cloud? Does rotation (vorticity) assist or resist collapse?

8. Consider a *hydrostatic* fluid in a constant external gravitational field with acceleration $-g\,\hat{z}$.

   (a) Suppose an object is immersed in the fluid. Show that the net force exerted on the object by the surrounding fluid is $M'g\,\hat{z}$, where M' is the mass of fluid displaced by the object.

   (b) Suppose the fluid, in addition to be hydrostatic, is also an *incompressible liquid* (e.g., water). Find the pressure $P(z)$ of the liquid.

9. Consider a *hydrostatic* spherical distribution of matter (e.g., a star, or a galaxy, or a cluster) having a gravitational potential

$$\Psi(r) = -\frac{GM_\star}{\sqrt{r^2 + b^2}},$$

where $M_\star$ and b are constants.

   (a) Find the density $\rho(r)$ for the system.

   (b) What is the mass $M(r)$ contained within a radius r? Find the total mass of the system.

   (c) Calculate the pressure $P(r)$ of the object. Show that it can be written as $P \propto \rho^\gamma$. What is the value of γ?

10. The Lane-Emden equation for polytropic stars is given by

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n$$

Consider the case of a polytrope with index $n = 1$.

   (a) By making the substitution $\chi(\xi) = \xi\theta(\xi)$, show that the Lane-Emden equation reduces to the simple harmonic oscillator equation:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\xi^2} + \chi = 0.$$

   (b) Apply the boundary conditions at the centre, $\theta(0) = 1$ and $\theta'(0) = 0$, to derive the analytic solution:

$$\theta(\xi) = \frac{\sin\xi}{\xi}$$

(c) Determine the location of the stellar surface ξ_1. Does a star with $n = 1$ have a finite radius?

11. It is known that a static solar corona dominated by *thermal conduction* cannot be confined because $P_\infty > P_{\rm ISM}$. Let us repeat this analysis for an *adiabatic* corona, where thermal conduction is negligible.

    (a) Assume the corona is an ideal gas obeying the adiabatic relation $P = K\rho^\gamma$ (with $\gamma = 5/3$). Combining this with the equation of hydrostatic equilibrium:

    $$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{GM_\odot}{r^2}$$

    show that the temperature varies with radius as:

    $$T(r) = T_0 - \frac{\gamma-1}{\gamma}\frac{\mu m_p}{k_B}GM_\odot\left(\frac{1}{R_\odot} - \frac{1}{r}\right)$$

    assuming $T(R_\odot) = T_0$. Here μ is the mean molecular weight.

    (b) Show that for an adiabatic corona, the temperature reaches zero at a finite radius $R_{\rm max}$.

    (c) Calculate this height $R_{\rm max}$ (in units of solar radii $R_\odot$) for $T_0 = 2 \times 10^6$ K. Take $\mu = 0.6$ for fully ionized solar plasma.

    (d) Unlike the conductive case, an adiabatic atmosphere *can* be static and confined (since $P \to 0$ at finite r). Why, then, do we still believe the solar wind exists?
    Hint: Consider the stability of such a steep temperature gradient against convection or conduction.