Cosmology
Lecture 15

Quasi-linear and non-linear perturbations

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Quasi-linear evolution

- We have worked out the linear theory of perturbations for both dark matter and for baryons. The next stage would be to calculate the evolution using the full non-linear theory.
- As it turns out, solving the equations in such a case becomes extremely non-trivial for both dark matter and baryons.
- A quasi-linear calculation can be performed in the case of dark matter. However, for the baryons, the pressure term complicate matters, and there is still no well-established rigorous quasi-linear or non-linear theory.
- Let us start with the Poisson equation and define a scaled potential
$$\psi \equiv \frac{2}{3H_0^2\Omega_{m,0}} \frac{a}{D}\phi,$$
so that the Poisson equation becomes
$$\vec{\nabla}^2\psi = \frac{\delta_{\rm DM}}{D}.$$
- Since $\delta_{\rm DM} \propto D$ in the linear theory, ψ has to be independent of time and hence $\psi(t, \vec{x}) = \psi_0(\vec{x})$. The field $\psi_0(\vec{x})$ can be thought of the initial potential field configuration.
- The actual potential ϕ is given by
$$\phi(t, \vec{x}) = \frac{D}{a} \frac{3H_0^2\Omega_{m,0}}{2} \psi_0(\vec{x}).$$
- Then the density contrast can be written as
$$\delta_{\rm DM}(t, \vec{x}) = D(t)\ \vec{\nabla}^2\psi_0(\vec{x}),$$
consistent with the expected growth.

Velocity evolution

- The linear continuity equation $\dot{\delta}_{\rm DM} + a^{-1}\vec{\nabla}\cdot\vec{v}_{\rm DM} = 0$ gives $\dot{D}\,\vec{\nabla}^2\psi_0(\vec{x}) + a^{-1}\vec{\nabla}\cdot\vec{v}_{\rm DM} = 0$, which implies $\vec{\nabla}\cdot\left(a\dot{D}\vec{\nabla}\psi_0 + \vec{v}_{\rm DM}\right) = 0$.
- Thus
$$\vec{v}_{\rm DM}(t,\vec{x}) = -a\dot{D}\vec{\nabla}\psi_0(\vec{x}) + \vec{v}_\perp(t,\vec{x}),$$
where
$$\vec{v}_\perp(t,\vec{x}) = \vec{\nabla}\times\vec{E}(t,\vec{x}),$$
with $\vec{E}$ being a vector field.
- Now, if we put this form of the velocity into the Euler equation, we obtain
$$\dot{\vec{v}}_\perp = -\frac{\dot{a}}{a}\,\vec{v}_\perp,$$
which has a solution
$$\vec{v}_\perp \propto \frac{1}{a}.$$
This is just the decay of vector perturbations.
- Since $\vec{v}_\perp$ decays with time (while the other part grows as $a\dot{D}$), we can write at late times
$$\vec{v}_{\rm DM}(t,\vec{x}) = -a\dot{D}\vec{\nabla}\psi_0(\vec{x}).$$

Zel'dovich approximation

- Since the Poisson equation is linear even in the most general case, we have $\vec{\nabla}^2\psi = -\delta_{\rm DM}/D$, which means that ψ does not evolve at all as long as $\delta_{\rm DM}$ is linear; the evolution begins only when $\delta_{\rm DM} \sim 1$.
- Since $\vec{v}_{\rm DM} = -a\,\dot{D}\vec{\nabla}\psi_0$ in the linear regime, the velocity usually evolves slower than the density contrast.
- For example, in the matter-dominated case, we have $\delta_{\rm DM} \propto a$ while $\vec{v}_{\rm DM} \propto a^{1/2}$. Hence in the quasi-linear regime, we can expect $\vec{v}_{\rm DM}$ to be linear even when the densities have become mildly non-linear.
- Since $\vec{v}_{\rm DM} = a\dot{\vec{x}}$, we get $\dot{\vec{x}} = -\dot{D}\vec{\nabla}\psi_0$ in the linear regime, which has a solution

  $$\vec{x}(t) = \vec{q} - D(t)\vec{\nabla}\psi_0(\vec{q}),$$

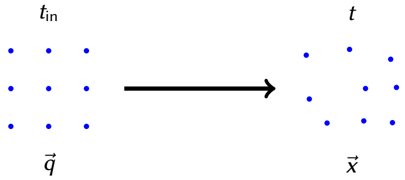
  with $\vec{q}$ being the initial position (in comoving units) of the fluid element.
- This is known as the **Zel'dovich approximation** which has been derived assuming essentially the linear theory.
- Zel'dovich suggested that the evolution of $\vec{x}$ can be extrapolated to situations where the displacements are not small any more.

Lagrangian coordinates

- In terms of components, the Zel'dovich solution looks like

$$x^a = q^a - D\,\frac{\partial \psi_0}{\partial q^a}, \quad a = 1, 2, 3.$$

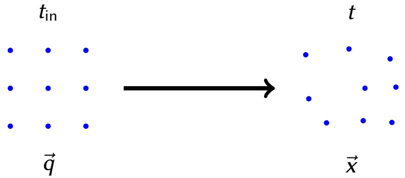
- So given an initial configuration $\vec{q}$ at $t = t_{\rm in}$, we can calculate $\delta_{\rm DM}(\vec{q})$ and hence can obtain ψ_0 using linear theory equation $\delta_{\rm DM}(\vec{q}) = D(t_{\rm in})\vec{\nabla}^2\psi_0(\vec{q})$.
- The above equation then gives all the particle positions and velocities at a later time which can be used for obtaining any quantity of interest.



- In fluid mechanics, the coordinates $\vec{q}$ are known as the **Lagrangian coordinates**, while $\vec{x}$ are known as the **Eulerian coordinates**.

Deformation tensor

- Now at $t > t_{\text{in}}$, mass conservation requires that $\rho_0[t, \vec{x}(t, \vec{q})]\mathrm{d}^3x = \rho_{\text{in}}(\vec{q})\mathrm{d}^3q$.



- The density field as a function of Lagrangian coordinates reads

$$\rho_0(t, \vec{q})\mathrm{d}^3x = \rho_{\text{in}}(\vec{q})\mathrm{d}^3q = \frac{\rho_{\text{in}}(\vec{q})}{\left|\frac{\partial x^a}{\partial q^b}\right|}\mathrm{d}^3x.$$

- Here the **deformation tensor** accounts for the gravitational evolution of the fluid

$$\frac{\partial x^a}{\partial q^b} = \frac{\partial q^a}{\partial q^b} - D\,\frac{\partial^2 \psi_0}{\partial q^a \partial q^b} = \delta^{ab} - D\,\frac{\partial^2 \psi_0}{\partial q^a \partial q^b}.$$

- At the linear stage, when $D\partial\psi_0/\partial q^a \ll 1$, the density relation can be approximated by

$$\rho_0(t, \vec{q}) = \frac{\rho_{\text{in}}(\vec{q})}{\left|\delta^{ab} - D\,\frac{\partial^2 \psi_0}{\partial q^a \partial q^b}\right|} \approx \rho_{\text{in}}(\vec{q})\left(1 + D\,\vec{\nabla}^2\psi_0\right),$$

giving $\delta \approx D\vec{\nabla}^2\psi_0$ and we recover the growing solution of the linear theory.

Pancakes, filaments and other structures

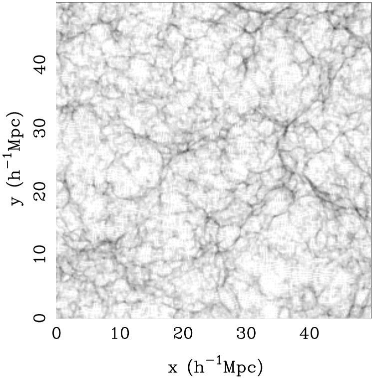
- The deformation tensor can be represented by a real symmetric matrix. It has three eigenvalues given by $1 - D\alpha(\vec{q}), 1 - D\beta(\vec{q}), 1 - D\gamma(\vec{q})$, where α, β, γ are the eigenvalues of the matrix $\partial^2\psi_0/\partial q^a \partial q^b$.
- The eigenvalues are the components of the deformation tensor along the three principal axes, i.e., the tensor becomes diagonal when the coordinate axes are rotated to be along the principal axes.
- The determinant is just the product of eigenvalues, hence

$$\rho_0(t, \vec{q}) = \frac{\rho_{\rm in}(\vec{q})}{[1 - D\alpha(\vec{q})][1 - D\beta(\vec{q})][1 - D\gamma(\vec{q})]}.$$

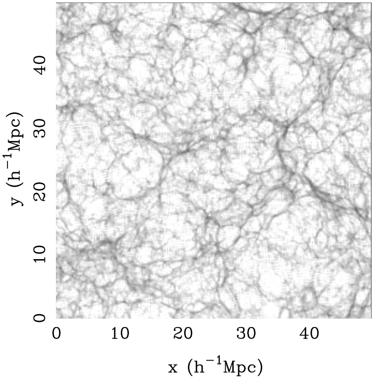
- If the eigenvalues are ordered in such a way that $\alpha(\vec{q}) \geq \beta(\vec{q}) \geq \gamma(\vec{q})$, then, as $D(t)$ grows, the first singularity occurs where α attains its maximum positive value $\alpha_{\rm max}$, at the time $D(t_1) = \alpha_{\rm max}^{-1}$. This corresponds to the formation of a "pancake" (sheet-like structure) by contraction along one of the principal axes.
- For this reason, Zel'dovich argued that pancakes are the first structures formed by gravitational clustering. Other structures like filaments and knots come from simultaneous contractions along two and three axes, respectively.
- The Zel'dovich approximation predicts the first non-linear structure to arise correspond to the high peaks of the $\alpha(\vec{q})$ field.
- However, within the Zel'dovich prescription, particles continue travelling along straight lines even after a pancake forms. In reality, we expect that the potential wells of the non-linear structures would not allow the particles to disperse. This approximation thus fails after the shells of matter start crossing each other.

Comparing with simulations

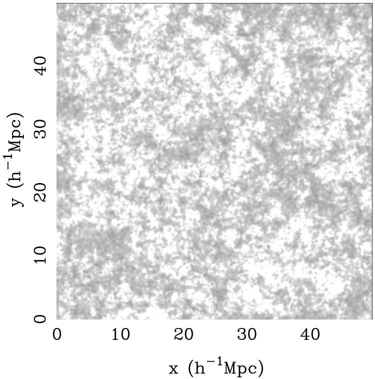
Non-linear



Zel'dovich approximation



Linear $\delta \propto D$



TRC, Haehnelt & Regan (2009)

Particle trajectories

- To understand the difficulties in working out a full non-linear theory of perturbations, let us work out the trajectories of the fluid elements (loosely called "particles") in full detail.
- The comoving coordinate is defined in terms of the physical coordinate as

$$\vec{r}(t) = a(t)\vec{x}(t) \quad \Longrightarrow \quad \vec{U} \equiv \dot{\vec{r}} = \frac{\dot{a}}{a}\vec{r} + a\dot{\vec{x}}$$

- Using the Newton force law

$$\ddot{\vec{r}} = -\vec{\nabla}_r \Phi,$$

  we get

$$\ddot{a}\vec{x} + 2\dot{a}\dot{\vec{x}} + a\ddot{\vec{x}} = -\vec{\nabla}_r \bar{\Phi} - \vec{\nabla}_r \phi.$$

- If $\vec{x} = \text{const}$, we have the equations corresponding to the smooth background, giving $\ddot{a}\vec{x} = -\vec{\nabla}_r \bar{\Phi}$.
- Hence the perturbed equation of motion is

$$\ddot{\vec{x}} + 2\frac{\dot{a}}{a}\dot{\vec{x}} = -\frac{1}{a^2}\vec{\nabla}\phi,$$

  This needs to be supplemented by the Poisson equation

$$\nabla^2 \phi = 4\pi G a^2 \bar{\rho}\delta.$$

- These equations can be solved for particle trajectories and the full non-linear evolution can be obtained, which is usually done in N-body numerical simulations.

Density and density contrast

- The (comoving) density is (assuming all to be of the same mass m)

$$\rho_0(t, \vec{x}) = m \sum_{i=1}^{N} \delta_D\left(\vec{x} - \vec{x}_i(t)\right),$$

  where $\vec{x}_i(t)$ is the trajectory of the ith particle.

- The average density is given by integrating over a large volume V

$$\bar{\rho}_0(t) = \int_V \frac{\mathrm{d}^3 x}{V}\, \rho_0(t, \vec{x}) = \frac{mN}{V}.$$

- Hence the density contrast is given by

$$\delta(t, \vec{x}) = \frac{V}{N} \sum_i \delta_D\left(\vec{x} - \vec{x}_i(t)\right) - 1.$$

- The Fourier transform of δ is given by

$$\delta(t, \vec{k}) = \int \mathrm{d}^3 x\, \delta(t, \vec{x})\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} = \frac{V}{N} \sum_i \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}_i(t)} - (2\pi)^3 \delta_D(\vec{k}).$$

Discrete to continuum

- The summation can be transformed into an integral by realizing that both the quantities V/N and $\mathrm{d}^3 q$ represent volume per particle, so

$$\frac{V}{N}\sum_i \longrightarrow \int \mathrm{d}^3 q.$$

- The density contrasts are then given by

$$\delta(t,\vec{x}) = \int \mathrm{d}^3 q \, \delta_D\left(\vec{x} - \vec{x}(t,\vec{q})\right) - 1,$$

and

$$\delta(t,\vec{k}) = \int \mathrm{d}^3 x \, \delta(t,\vec{x}) \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} = \int \mathrm{d}^3 q \, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}(t,\vec{q})} - (2\pi)^3 \delta_D(\vec{k}).$$

The derivatives of δ

- To find the differential equation satisfied by $\delta(\vec{k})$, let us calculate the derivatives

$$\dot{\delta}(\vec{k}) = \int \mathrm{d}^3 q \, \left[-\mathrm{i}\vec{k} \cdot \dot{\vec{x}}(\vec{q})\right] \, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}(\vec{q})},$$

$$\begin{aligned}
\ddot{\delta}(\vec{k}) &= \int \mathrm{d}^3 q \left[-\mathrm{i}\vec{k} \cdot \ddot{\vec{x}}(\vec{q}) - \left(\vec{k} \cdot \dot{\vec{x}}(\vec{q})\right)^2\right] \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}(\vec{q})} \\
&= \int \mathrm{d}^3 q \left[-\mathrm{i}\vec{k} \cdot \left(-2\frac{\dot{a}}{a}\dot{\vec{x}}(\vec{q}) - \frac{1}{a^2}\vec{\nabla}\phi\right) - \left(\vec{k} \cdot \dot{\vec{x}}(\vec{q})\right)^2\right] \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}(\vec{q})} \\
&= -2\frac{\dot{a}}{a}\dot{\delta}(\vec{k}) + \frac{1}{a^2}\int \mathrm{d}^3 q \, \mathrm{i}\vec{k} \cdot \vec{\nabla}\phi \, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}(\vec{q})} - \int \mathrm{d}^3 q \left(\vec{k} \cdot \dot{\vec{x}}(\vec{q})\right)^2 \, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}(\vec{q})}.
\end{aligned}$$

- To evaluate the second term on the right, we have to calculate ϕ in terms of the trajectories or density.

The Poisson equation

- The Poisson equation
$$\vec{\nabla}^2 \phi = \frac{3}{2} \frac{H_0^2}{a} \Omega_{m,0} \delta,$$
has the solution (recall electrostatics)
$$\phi[\vec{x}(\vec{q})] = -\frac{3 H_0^2 \Omega_{m,0}}{8 \pi a} \int \mathrm{d}^3 x' \, \frac{\delta(\vec{x})}{|\vec{x}(\vec{q}) - \vec{x}'|}.$$
- This can be written in terms of the Fourier transform as
$$\phi[\vec{x}(\vec{q})] = -\frac{3 H_0^2 \Omega_{m,0}}{2a} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\vec{k}' \cdot \vec{x}(\vec{q})}}{k'^2} \, \delta(\vec{k}').$$
- Hence we can compute the relevant quantities
$$\vec{\nabla} \phi = -\frac{3 H_0^2 \Omega_{m,0}}{2a} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \, \mathrm{i}\vec{k}' \, \frac{\mathrm{e}^{\mathrm{i}\vec{k}' \cdot \vec{x}(\vec{q})}}{k'^2} \delta(\vec{k}')$$
$$\frac{1}{a^2} \int \mathrm{d}^3 q \, \mathrm{i}\vec{k} \cdot \vec{\nabla} \phi \, \mathrm{e}^{-\mathrm{i}\vec{k} \cdot \vec{x}(\vec{q})} = \frac{3 H_0^2 \Omega_{m,0}}{2a^3} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{\vec{k} \cdot \vec{k}'}{k'^2} \delta(\vec{k} - \vec{k}') \delta(\vec{k}') + \frac{3 H_0^2 \Omega_{m,0}}{2a^3} \delta(\vec{k}).$$

The differential equation for non-linear perturbations

- We can collect all the terms and write

$$\ddot{\delta}(\vec{k}) + 2\frac{\dot{a}}{a}\dot{\delta}(\vec{k}) = \frac{3H_0^2\Omega_{m,0}}{2a^3}\delta(\vec{k}) + A(\vec{k}) - B(\vec{k}),$$

  where the non-linear terms are given by

$$A(\vec{k}) = \frac{3H_0^2\Omega_{m,0}}{2a^3}\,\vec{k}\cdot\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{\vec{k}'}{k'^2}\,\delta(\vec{k}-\vec{k}')\,\delta(\vec{k}'),$$

  and

$$B(\vec{k}) = \int\mathrm{d}^3q\left[\vec{k}\cdot\dot{\vec{x}}(\vec{q})\right]^2\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}(\vec{q})}.$$

- The first term shows that different $\vec{k}$-modes get coupled as the system becomes non-linear.
- The second term depends of $\vec{x}(\vec{q})$ and its derivative, hence the equation is not closed.
- Both the terms become sub-dominant in the linear regime, and also when $|\vec{k}|\to 0$.
- The forms of the non-linear terms can be worked out when there is some symmetry in the problem.

Homogeneous trajectories

- Consider a set of trajectories which are homogeneous (i.e., the initial shape distribution is preserved)

$$\vec{x}(\vec{q}, t) = f(t)\vec{q},$$

where $f(t)$ is a function which is to be determined.

- It can be related to the density contrast using

$$\delta(t, \vec{k}) = \int \mathrm{d}^3 q \, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}(t,\vec{q})} - (2\pi)^3 \delta_D(\vec{k}) = (2\pi)^3 \delta_D(\vec{k}) \left(\frac{1}{f^3} - 1\right).$$

- Also

$$\delta(t, \vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \delta(t, \vec{k}) \, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} = \frac{1}{f^3} - 1 \equiv \delta(t).$$

- So the density contrast only has a $\vec{k} = 0$ mode,

$$\delta(\vec{k}, t) = (2\pi)^3 \delta_D(\vec{k}) \delta(t).$$

- The non-linear terms can be computed easily

$$A(\vec{k}) = \frac{3H_0^2 \Omega_{m,0}}{2a^3} \, \delta^2(t) \, (2\pi)^3 \delta_D(\vec{k}), \qquad B(\vec{k}) = -(2\pi)^3 \delta_D(\vec{k}) \, \frac{4}{3} \frac{\dot{\delta}^2}{(1+\delta)}.$$

- So the equation of motion is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \frac{3H_0^2 \Omega_{m,0}}{2a^3}\delta + \frac{3H_0^2 \Omega_{m,0}}{2a^3}\delta^2 + \frac{4}{3}\frac{\dot{\delta}^2}{(1+\delta)}.$$

Spherical evolution

- As an example of a homogeneous system, consider a spherical region of mass M having a proper radius $R(t)$ which is evolving under gravity. The comoving density is given by

$$\rho_0(t) = \frac{1}{a^3} \frac{3M}{4\pi R^3(t)}.$$

- The evolution of the background matter density and the contrast are given by

$$\bar{\rho}_0(t) = \frac{\Omega_{m,0} 3 H_0^2}{8\pi G} \quad \Longrightarrow \quad 1 + \delta(t) = \frac{2GM}{\Omega_{m,0} H_0^2} \frac{a^3(t)}{R^3(t)}.$$

  Note that $f(t) \propto a(t)/R(t)$.
- To get the equation of motion, we compue

$$\dot{\delta} = 3(1+\delta)\left(\frac{\dot{a}}{a} - \frac{\dot{R}}{R}\right), \qquad \ddot{\delta} = 3(1+\delta)\left(2\frac{\dot{a}^2}{a^2} - 6\frac{\dot{a}}{a}\frac{\dot{R}}{R} + 4\frac{\dot{R}^2}{R^2} + \frac{\ddot{a}}{a} - \frac{\ddot{R}}{R}\right).$$

- Use the Friedmann equation (including also any other component besides non-relativistic matter)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\bar{\rho} + \bar{\rho}_{\text{rest}} + 3\bar{P}_{\text{rest}}\right)$$

  to get

$$\ddot{R} = -\frac{GM}{R^2} - \frac{4\pi G}{3}(\bar{\rho}_{\text{rest}} + 3\bar{P}_{\text{rest}})R.$$

- This determines the evolution of a spherical system. The first term on the right is simply the Newtonian acceleration.