Cosmology
Lecture 13
CMB anisotropies

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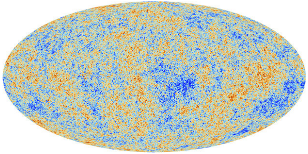
Temperature fluctuations

- The CMB sky shows small fluctuations in the temperature, i.e., the value is different in different directions

$$\Theta(\hat{n}) \equiv \frac{\delta T(\hat{n})}{T_0} \sim 10^{-5}.$$



- Analogous to Fourier transforms, one can expand a field on a spherical surface as

$$\Theta(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}\, Y_{\ell m}(\hat{n}), \quad a_{\ell m} = \int \mathrm{d}\Omega\, \Theta(\hat{n})\, Y^*_{\ell m}(\hat{n}).$$

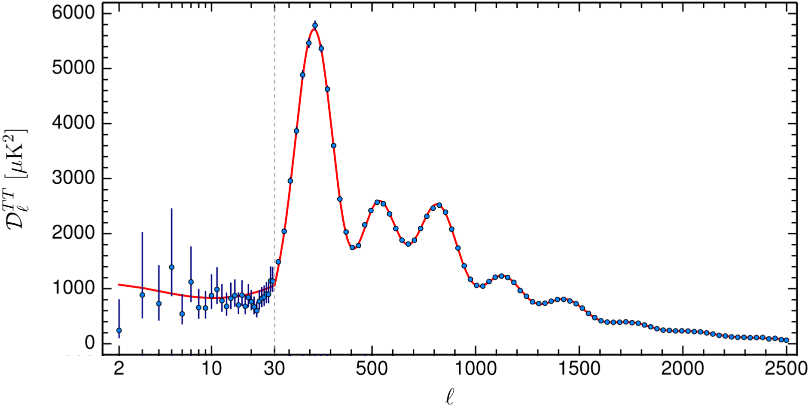
- The equivalent of the power spectrum is the angular power spectrum (to be defined more rigorously later)

$$\mathcal{C}_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2 .$$

Sometimes the power spectrum is defined as $T_0^2\, \mathcal{C}_\ell$. Also, $\mathcal{D}_\ell = \ell(\ell + 1)\dfrac{\mathcal{C}_\ell}{2\pi}$.

Features in the angular power spectrum

- The measured power spectrum agrees extremely well with the theoretical one. Shows various features at different angular scales.



- We will discuss the physics of these features in a simplistic manner.

Radiation perturbations

- To the zeroth order, we assume that the temperature fluctuations measure the radiation perturbations at the last scattering surface (keeping in mind that $\rho_r \propto T^4$)

$$\Theta(\hat{n}) \sim \frac{1}{4}\delta_r(\eta_{\rm LSS}, \vec{x} = \eta_{\rm LSS}\hat{n})$$

- The evolution of the different species is given by the conservation equations (with $k^2 \equiv \gamma^{\alpha\beta} k_\alpha k_\beta$)

$$\delta' = (k^2 V + 3\phi')\left(1 + \frac{\bar{P}}{\bar{\rho}}\right) - 3\frac{a'}{a}\,\frac{p - \bar{P}\delta}{\bar{\rho}},$$

$$V' = -4\frac{a'}{a}V - \frac{\bar{\rho}' + \bar{P}'}{\bar{\rho} + \bar{P}}V - \frac{1}{\bar{\rho} + \bar{P}}p - \phi.$$

- For radiation, we have $\bar{P}_r = \bar{\rho}_r/3$. If we assume $P_r = \rho_r/3$, then $p_r = P_r - \bar{P}_r = \bar{P}_r\delta_r$.
- Also, $\bar{\rho}_r \propto a^{-4} \implies \bar{\rho}_r'/\bar{\rho}_r = -4a'/a$. The equations are then

$$\delta_r' = \frac{4}{3}k^2 V_r + 4\phi', \qquad V_r' = -\frac{\delta_r}{4} - \phi.$$

- For baryons, we can ignore the pressure, use $\bar{\rho}_m'/\bar{\rho}_m = -3a'/a$ and write

$$\delta_b' = k^2 V_b + 3\phi', \qquad V_b' = -\frac{a'}{a}V_b - \phi.$$

  Note that the same equations should hold for dark matter too.

Baryon-photon fluid

- The equations we wrote hold when the different fluids are not interacting with each other. However, we know that before decoupling, the photons and baryons (electrons) were scattering off each other.
- In that case, we need to add the interaction term to the equations. Without deriving the term, we write it directly as

$$\delta_r' = \frac{4}{3} k^2 V_r + 4\phi', \quad V_r' = -\frac{\delta_r}{4} - \phi - a\sigma_T n_e \left(V_r - V_b\right),$$
$$\delta_b' = k^2 V_b + 3\phi', \quad V_b' = -\frac{a'}{a} V_b - \phi + a\sigma_T n_e \frac{V_r - V_b}{R},$$

  where

$$R \equiv \frac{3\bar{\rho}_b}{4\bar{\rho}_r}.$$

- The interaction term acts as a "drag" and tries to bring the baryons and photons to have the same velocity.
- In fact, the (comoving) photon mean free path $\lambda_T = (a\sigma_T n_e)^{-1} \sim 1 - 2$ Mpc just before decoupling. Compare this with the comoving Hubble radius $\eta = (aH)^{-1} \gtrsim 200$ Mpc. Thus we can take the limit $\lambda_T \to 0$, which is known as the **tight-coupling approximation**.
- To the zeroth order in λ_T, we have

$$V_r \approx V_b, \quad \delta_r \approx \frac{4}{3}\delta_b.$$

First order corrections

- To obtain the first order corrections to the tight-coupling limit, we use the zeroth order solutions and write the baryon velocity equation as

$$a\sigma_T n_e(V_r - V_b) = R\left(V_b' + \frac{a'}{a}V_b + \phi\right) \approx R\left(V_r' + \frac{a'}{a}V_r + \phi\right)$$

- Put this in the photon velocity equation to obtain the first order corrections

$$\delta_r' = \frac{4}{3}k^2 V_r + 4\phi', \quad V_r' = -\frac{a'}{a}\frac{R}{1+R}V_r - \frac{1}{4(1+R)}\delta_r - \phi.$$

- Use $R'/R = \bar{\rho}_b'/\bar{\rho}_b - \bar{\rho}_r'/\bar{\rho}_r = a'/a$, and eliminate V_r from the two equations

$$\delta_r'' + \frac{R'}{1+R}\delta_r' + k^2 c_s^2 \delta_r = G$$

where

$$c_s = \frac{1}{\sqrt{3(1+R)}}, \quad G = 4\left(\phi'' + \frac{R'}{1+R}\phi' - \frac{1}{3}k^2\phi\right).$$

- Note that the square of the baryon-photon sound speed is

$$\frac{\partial P}{\partial \rho} = \frac{P_r'}{\rho_b' + \rho_r'} = \frac{1}{3}\frac{\rho_r'}{\rho_b' + \rho_r'} = \frac{1}{3}\left[\frac{-4(a'/a)\rho_r}{-3(a'/a)\rho_b - 4(a'/a)\rho_r}\right] = \frac{1}{3}\frac{1}{1+R}.$$

Thus the sound speed is nothing but c_s.

Approximate solutions

- The equation has the form of a forced harmonic oscillator

$$\delta_r'' + \frac{R'}{1+R}\delta_r' + k^2 c_s^2 \delta_r = G.$$

  The solutions will be oscillatory with a phase $\varphi = k \int \mathrm{d}\eta\ c_s(\eta)$.

- We can define the **sound horizon** as

$$r_s(\eta) \equiv \int_0^\eta \mathrm{d}\eta'\ c_s(\eta') = \frac{1}{\sqrt{3}} \int_0^\eta \frac{\mathrm{d}\eta'}{\sqrt{1+R(\eta')}}.$$

- If the phase of the solution varies faster than the amplitude, we can use WKB approximation to find the solution to the homogeneous equation. This condition is effectively $k\, c_s \gg \dot{R}/(1+R) \sim \eta^{-1}$, i.e., valid for scales smaller than the sound horizon.

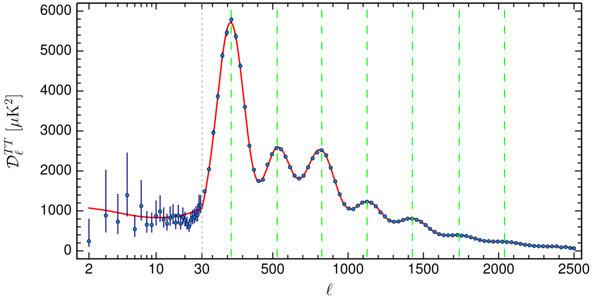
- Once the solutions to the homogeneous part are found, one can use the Wronskian to find the full solution. The solution in this approximation turns out to be

$$\delta_r(\eta) \approx A_1 \frac{\sin(k r_s)}{[1+R(\eta)]^{1/4}} + A_2 \frac{\cos(k r_s)}{[1+R(\eta)]^{1/4}} + \frac{\sqrt{3}}{k} \int_0^\eta \mathrm{d}\eta' \frac{[1+R(\eta')]^{3/4}}{[1+R(\eta)]^{1/4}} \sin\left[k \left\{ r_s(\eta) - r_s(\eta') \right\}\right]\ G(\eta').$$

  The constants can be fixed by the initial conditions.

Acoustic peaks

- The solution clearly has peaks and troughs with a scale r_s. The power spectrum $\propto |\delta_r|^2$ will have peaks separated by $\Delta k_s = \pi / r_s$.
- The corresponding separation in angular multipoles would be $\Delta \ell_s = \eta_{\mathrm{LSS}} \, \Delta k_s = \pi \eta_{\mathrm{LSS}} / r_s$. The corresponding angular scale is $\theta_s = \pi / \Delta \ell_s = r_s / \eta_{\mathrm{LSS}}$.
- These oscillatory features in the power spectrum are called **baryon acoustic oscillations (BAO)**.
- Observations can constrain θ_s (which depends on the cosmological parameters).

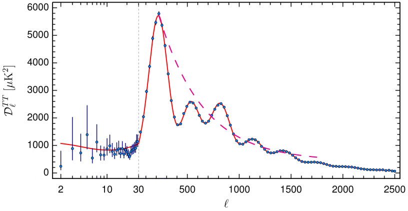


Diffusion damping

- To the second order in $\lambda_T \equiv (a\sigma_T n_e)^{-1}$, one finds that the photons diffuse from hot (high density) regions to cold (low density). This leads to a decrease in fluctuations, called **diffusion damping** or **Silk damping**.
- To work out the details, it turns out that the simple perfect fluid picture for photons is not sufficient. Rather one has to work with the Boltzmann equation involving the distribution function $f(\eta, \vec{p}, \vec{x})$.
- An order of magnitude estimate is possible for the diffusion damping scale λ_D. If a photon does a random walk and suffers $N_{\rm scat}$ scatterings off electrons, it travels a distance $\lambda_D \sim \sqrt{N_{\rm scat}}\, \lambda_T$. Over a cosmic time η, the number of scatterings would be $N_{\rm scat} \sim \eta / \lambda_T$. Thus $\lambda_D^2 \sim \eta\, \lambda_T$.
- At scales $\ll \lambda_D$, the fluctuations will be damped.
- The full calculation shows

$$\left(\frac{\lambda_D}{2\pi}\right)^2 \equiv \frac{1}{k_D^2} = \frac{1}{6} \int_0^\eta \mathrm{d}\eta' \, \frac{\lambda_T(\eta')}{[1 + R(\eta')]^2} \left[R^2(\eta') + \frac{4}{5 f_P} \left(1 + R(\eta')\right)\right],$$

  where f_P is a correction factor for polarization ($= 1$ for isotropic unpolarized light).
- This leads to $\delta_r \longrightarrow \delta_r\, \mathrm{e}^{-k^2/k_D^2}$.



BAO post decoupling

- After the baryons decouple from the radiation, the perturbations grow $\delta_b \propto a$. Similarly, $\delta_{\rm DM} \propto a$ and $\bar{\rho}_m \delta_m = \bar{\rho}_{\rm DM} \delta_{\rm DM} + \bar{\rho}_b \delta_b$.
- The growth of δ_b contain both density and velocity perturbations from the decoupling epoch which leads to shifts in the BAO peaks.
- In presence of baryons, the matter transfer function changes to

$$T(k) = \frac{\Omega_{{\rm DM},0}}{\Omega_{m,0}} T_{\rm DM}(k) + \frac{\Omega_{b,0}}{\Omega_{m,0}} T_b(k), \quad \Omega_{m,0} = \Omega_{{\rm DM},0} + \Omega_{b,0}$$

  The effects of the BAO would be imprinted through $T_b(k)$.

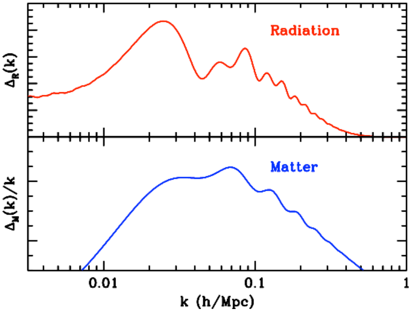


Figure courtesy Martin White

BAO in galaxy surveys

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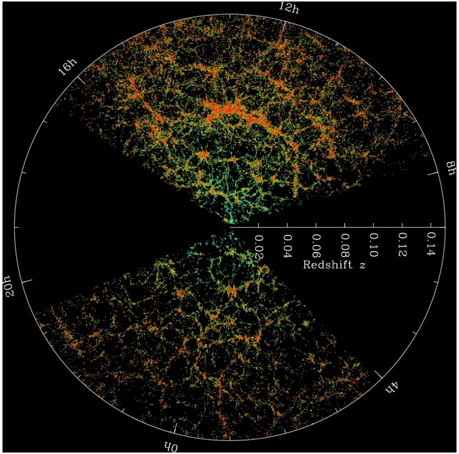
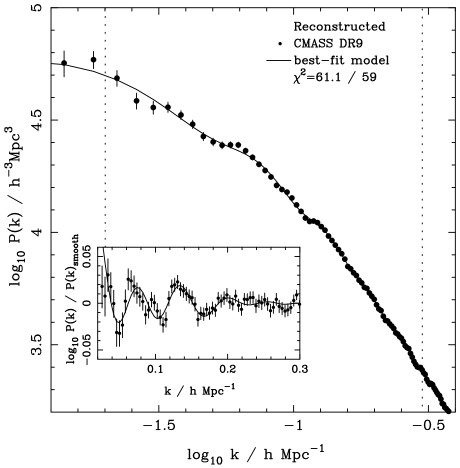




Anderson et al (2012)

Boltzmann equation for photons

- In general, the evolution of radiation perturbations need to be solved using the distribution function $f(\eta, \vec{p}, \vec{x})$, which is the number per unit phase space volume. The distribution satisfies the Boltzmann equation
$$\frac{\partial f}{\partial \eta} + \frac{\mathrm{d}\vec{p}}{\mathrm{d}\eta} \cdot \vec{\nabla}_p f + \frac{\mathrm{d}\vec{x}}{\mathrm{d}\eta} \cdot \vec{\nabla} f = C[f],$$
where the right hand side contains the effects from collisions (i.e., absorption, emission, scattering).
- This needs to be supplemented by the photon geodesic equation
$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}\eta} = p\,\vec{\nabla}(\phi + \psi), \quad \frac{\mathrm{d}\vec{x}}{\mathrm{d}\eta} = -(1 + \phi + \psi)\frac{\vec{p}}{p} = -(1 + \phi + \psi)\hat{n}.$$
The photon energy is $E = (1 + \phi)p/a$.
- If we assume that the perturbations leave the form of the photon distribution as blackbody and only change the value of the temperature, we write
$$f(\eta, \vec{p}, \vec{x}) = \bar{f} + \delta f = \bar{f}\left(\frac{E}{T_0 + \delta T(\eta, \hat{n}, \vec{x})}\right).$$
- Defining $\Theta(\eta, \hat{n}, \vec{x}) \equiv \delta T/T_0$, we can write the Boltzmann equation as
$$\frac{\partial \Theta}{\partial \eta} + \hat{n} \cdot \vec{\nabla}\Theta + \hat{n} \cdot \vec{\nabla}\psi - \frac{\partial \phi}{\partial \eta} = a\sigma_T n_e\left[\Theta_0 - \Theta + \hat{n} \cdot \vec{\nabla} V_b + \frac{1}{16}\Pi_{\alpha\beta} n^\alpha n^\beta\right],$$
with
$$\Theta_0(\eta, \vec{x}) \equiv \int \frac{\mathrm{d}\Omega}{4\pi}\Theta(\eta, \hat{n}, \vec{x}) = \frac{1}{4}\delta_r, \quad \Pi^{\alpha\beta} \equiv 12\int \frac{\mathrm{d}\Omega}{4\pi}\Theta\left(n^\alpha n^\beta - \frac{1}{3}\delta^{\alpha\beta}\right).$$
$\Pi^{\alpha\beta}$ is the anisotropic stress which arises from the angular dependence of the Thomson scattering.
- Note that a non-zero $\Pi^{\alpha\beta}$ makes $\phi \neq \psi$.

Fluid equations

- Let us write the Boltzmann equation in Fourier space

$$\frac{\partial \Theta}{\partial \eta} + \mathrm{i}\hat{n}\cdot\vec{k}\Theta + \mathrm{i}\hat{n}\cdot\vec{k}\psi - \frac{\partial \phi}{\partial \eta} = a\sigma_T n_e \left[\Theta_0 - \Theta + \mathrm{i}\hat{n}\cdot\vec{k}V_b + \frac{1}{16}\Pi_{\alpha\beta}n^\alpha n^\beta\right].$$

- Taking the zeroth moment, i.e., $\int \mathrm{d}\Omega/4\pi \times (\ldots)$ gives the continuity equation

$$\delta_r' = \frac{4}{3}k^2 V_r + 4\phi',$$

  where the fluid bulk velocity is

$$\vec{v}_r = 3\int \frac{\mathrm{d}\Omega}{4\pi}\hat{n}\,\Theta \Longrightarrow \int \frac{\mathrm{d}\Omega}{4\pi} n_\alpha\,\Theta = \frac{\mathrm{i}}{3}k_\alpha V_r.$$

- The first moment, i.e., $\int \mathrm{d}\Omega/4\pi\ n_\alpha \times (\ldots)$ gives the Euler equation

$$V_r' = -\frac{\delta_r'}{4} - \psi - \frac{1}{4}\frac{k_\alpha k_\beta \Pi^{\alpha\beta}}{k^2} - a\sigma_T n_e\left(V_r - V_b\right).$$

  Note the presence of $\Pi^{\alpha\beta}$ in the equation, which becomes important in the second order tight-coupling limit.
- To close the system of equations, one needs an evolution equation for $\Pi^{\alpha\beta}$, which would involve higher order moments. This leads to a infinite series of equations called **Boltzmann hierarchy**.

Integral solution

- An alternate way to solve the equation is to write it as

$$\frac{\partial\left(\Theta+\psi\right)}{\partial\eta}+\hat{n}\cdot\vec{\nabla}\left(\Theta+\psi\right)+a\sigma_T n_e\left(\Theta+\psi\right)=\frac{\partial\left(\phi+\psi\right)}{\partial\eta}+a\sigma_T n_e S,\quad S\equiv\frac{\delta_r}{4}+\psi+\hat{n}\cdot\vec{\nabla}V_b+\frac{1}{16}\Pi_{\alpha\beta}n^{\alpha}n^{\beta}.$$

- This can be solved along a line of sight using the *method of characteristics*

$$\Theta(\eta_0)+\psi(\eta_0)=\int_0^{\eta_0}\mathrm{d}\eta'\ \mathrm{e}^{-\tau(\eta')}\frac{\partial\left(\phi+\psi\right)}{\partial\eta'}+\int_0^{\eta_0}\mathrm{d}\eta'\ g(\eta')\ S(\eta'),$$

where we have already defined the optical depth and visibility function, respectively, as

$$\tau(\eta)=\int_{\eta}^{\eta_0}\mathrm{d}\eta'\,a(\eta')\sigma_T n_e(\eta'),\quad g(\eta)=a(\eta)\sigma_T n_e(\eta)\ \mathrm{e}^{-\tau(\eta)}.$$

- The form of the solution tells us that Θ are sourced not only by δ_r, but also $\phi,\psi,\vec{\nabla}V_b$ (or $\vec{\nabla}V_r$) and so on.
- We can Fourier transform the solution and expand it as

$$\Theta(\eta_0,\hat{n},\vec{k})=4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-\mathrm{i})^{\ell}\ \Theta_{\ell}(\eta_0,\vec{k})\ Y_{\ell m}(\hat{n})\ Y^*_{\ell m}(\hat{k}).$$

The spherical harmonic coefficients are

$$a_{\ell m}=4\pi(-\mathrm{i})^{\ell}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Theta_{\ell}(\eta_0,\vec{k})\ Y^*_{\ell m}(\hat{k}).$$

Sachs-Wolfe effect

- At large scales $k \ll \eta_{\rm LSS}^{-1}$, one can obtain simple solutions. At super-Hubble scales, we ignore Π and $\vec{\nabla} V_b \to \vec{k} V_b$, and take $\psi = \phi =$ const.
- At $k\eta \to 0$ limit, the density perturbations remain constant $\delta_m = \delta_b = -2\phi$ and $\delta_r = 4\delta_b/3$.
- The line of sight solution becomes

$$\Theta(\eta_0, \hat{n}, \vec{x}) + \psi(\eta_0, \vec{x}) \approx \int_0^{\eta_0} \mathrm{d}\eta' \; g(\eta') \left[\frac{\delta_r\left(\eta', \vec{x} - \hat{n}(\eta_0 - \eta')\right)}{4} + \psi\left(\eta', \vec{x} - \hat{n}(\eta_0 - \eta')\right) \right]$$

$$\approx \int_0^{\eta_0} \mathrm{d}\eta' \; g(\eta') \left[\frac{\phi\left(\eta', \vec{x} - \hat{n}(\eta_0 - \eta')\right)}{3} \right].$$

- The visibility function peaks at $\eta = \eta_{\rm LSS}$ and we can take it as a delta function. One can then show that

$$\Theta_\ell(\eta_0, \vec{k}) = \frac{1}{3} \phi(\eta_{\rm LSS}, \vec{k}) \; j_\ell[k(\eta_0 - \eta_{\rm LSS})].$$

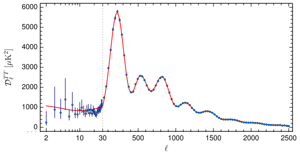
- Now, $C_\ell \propto |a_{\ell m}|^2 \propto |\Theta_\ell|^2 \propto |\phi(\vec{k})|^2 \propto P_\phi(k)$.
- If we take $k^3 P_\phi(\eta_{\rm LSS}, k)/2\pi^2 = A_s k^{n-1}$ with $n \approx 1$, we get

$$C_\ell = \frac{2}{9\pi} \int_0^\infty \frac{\mathrm{d}k}{k} \left[k^3 P_\phi(\eta_{\rm LSS}, k) \right] \left\{ j_\ell[k(\eta_0 - \eta_{\rm LSS})] \right\}^2 = \frac{4\pi^2}{9} A_s \frac{2^{n-4} \Gamma(3-n) \Gamma\left[(2\ell + n - 1)/2\right]}{\Gamma^2\left[(4-n)/2\right] \Gamma\left[(2\ell - n + 5)/2\right]}.$$

- For $n = 1$, it simplifies to

$$\frac{\ell(\ell+1) C_l}{2\pi} = \frac{A_s}{9}.$$

Thus it is a constant at small ℓ. This is a strong probe of the amplitude of $P_m(k)$.



Other CMB effects

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- The Thomson scattering also produces polarization which can be observed. The signal can be calculated and compared with obervations.
- The CMB picks up features from the post-recombination universe as it free streams to us, known as **secondary anisotropies** (as opposed to **primary anisotropies** generated during recombination). Some examples are
  - **Integrated Sachs-Wolfe effect:** The $\phi' + \psi'$ term in the solution, important only in the dark energy dominated era.
  - **Reionization:** The reionization by the first stars modifies the visibility function and generates additional signal (both temeprature and polarization).
  - **Lensing:** The CMB gets lensed by structures at low redshifts.
- The CMB also suffers **spectral distortions** where it departs from the blackbody curve. This arises, e.g., from the **Sunyaev-Zel'dovich effect** in hot clusters (CMB photons getting scattered off high-energy electrons) and 21 cm absorption/emission of neutral hydrogen.
- CMB has been the most important probe to constrain cosmological parameters, leading to the *standard model of cosmology*.

Parameter	TT+lowE 68% limits	TT,TE,EE+lowE+lensing+BAO 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02242 ± 0.00014
$\Omega_c h^2$	0.1206 ± 0.0021	0.11933 ± 0.00091
$100\theta_{MC}$	1.04077 ± 0.00047	1.04101 ± 0.00029
τ	0.0522 ± 0.0080	0.0561 ± 0.0071
$\ln(10^{10} A_s)$	3.040 ± 0.016	3.047 ± 0.014
n_s	0.9626 ± 0.0057	0.9665 ± 0.0038
H_0 [km s^{-1} Mpc^{-1}]	66.88 ± 0.92	67.66 ± 0.42
Ω_Λ	0.679 ± 0.013	0.6889 ± 0.0056
Ω_m	0.321 ± 0.013	0.3111 ± 0.0056
$\Omega_m h^2$	0.1434 ± 0.0020	0.14240 ± 0.00087
$\Omega_m h^3$	0.09589 ± 0.00046	0.09635 ± 0.00030
σ_8	0.8118 ± 0.0089	0.8102 ± 0.0060
$S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$	0.840 ± 0.024	0.825 ± 0.011
$\sigma_8 \Omega_m^{0.25}$	0.611 ± 0.012	0.6051 ± 0.0058
z_{re}	7.50 ± 0.82	7.82 ± 0.71

Planck Collaboration (2018)