Cosmology
Lecture 12

Transfer function of matter fluctuations

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Matter perturbations

- The perturbations of non-relativistic matter (dark matter and baryons) play the most important role in structure formation. These perturbations originate during the inflation, evolve during the radiation and matter dominated epochs and then finally through the cosmological constant dominated accelerating universe. We will ignore the accelerating universe for the time being.
- Let us first work out the evolution in the matter dominated regime. In this regime, we will ignore the radiation.
- Using $\bar{P}_m = 0$, it is easy to show that $a \propto \eta^2$. This can be shown if one remembers that $a \propto t^{2/3}$.
- We will assume that the perturbed pressure $p_m = 0$. This is a good approximation for dark matter which remain collisionless even at small scales.
- We can then show that the solutions to the $^{\alpha}_{\alpha}$ component is

$$\phi = C_1 + \frac{C_2}{\eta^5},$$

  implying that the potential, at best, can remain constant. The decaying mode dies rapidly with time.
- Note that ϕ remains a constant *at all scales*, as long as the expansion is driven by matter.

Evolution at large and small scales

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- Now, let us use the Poisson equation

$$\left(k^2 + \frac{12}{\eta^2}\right)\phi = -4\pi G a^2 \bar{\rho}_m \delta_m \propto -\frac{\delta_m}{a}.$$

- For large scales, we ignore the k^2 term and get

$$\frac{\phi}{\eta^2} \propto -\frac{\delta_m}{a} \Longrightarrow \delta_m \propto -\phi,$$

  which implies that δ_m does not evolve at large scales (scales larger than the Hubble radius).
- For small scales, we have

$$k^2\phi \propto -\frac{\delta_m}{a} \Longrightarrow \delta_m \propto -ak^2\phi,$$

  which implies that the perturbations grow as $\delta_m \propto a$.

Newtonian limit

- Before proceeding further, let us work out the equations at small scales $|k_\alpha|\eta \to \infty$:

$$\delta'_m = \gamma^{\alpha\beta} k_\alpha k_\beta V_m, \qquad V'_m = -4\frac{a'}{a}V_m - \frac{\bar{\rho}'}{\bar{\rho}}V_m - \frac{p_m}{\bar{\rho}_m} - \phi.$$

- Now for non-relativistic matter, we have $\bar{\rho}'_m/\rho_m = -3a'/a$. Also, let us transform to the usual time coordinate $\mathrm{d}t = \mathrm{d}\eta\, a(\eta)$. Then $\delta' = \partial\delta/\partial\eta = a\partial\delta/\partial t = a\,\dot{\delta}$ and hence

$$a\,\dot{\delta}_m = \gamma^{\alpha\beta} k_\alpha k_\beta V_m, \qquad a\dot{V}_m = -\dot{a}V_m - \frac{p_m}{\bar{\rho}_m} - \phi.$$

- Now write in terms of the vector $\vec{v}_m$

$$\dot{\delta}_m = -\frac{1}{a}\mathrm{i}k^\alpha v_{m,\alpha}, \qquad \dot{v}_{m,\alpha} = -\frac{\dot{a}}{a}v_{m,\alpha} - \frac{1}{a}\frac{\mathrm{i}k_\alpha p_m}{\bar{\rho}_m} - \frac{1}{a}\mathrm{i}k_\alpha\phi,$$

- If we transform back to real space $\mathrm{i}k_\alpha \to \partial/\partial x^\alpha$, we get

$$\dot{\delta}_m + \frac{1}{a}\vec{\nabla}\cdot\vec{v}_m = 0, \qquad \dot{\vec{v}}_m + \frac{\dot{a}}{a}\vec{v}_m = -\frac{1}{a}\frac{\vec{\nabla}p_m}{\bar{\rho}_m} - \frac{1}{a}\vec{\nabla}\psi.$$

  These are the evolution of perturbations on small scales, which can also be derived from the equations of fluid dynamics.
- In this small-scale limit, the ${}^0{}_0$ component of the Einstein equation becomes the Poisson equation

$$-k^\alpha k_\alpha\phi = 4\pi G a^2\bar{\rho}_m\delta_m \implies \vec{\nabla}^2\phi = 4\pi G a^2\bar{\rho}_m\delta_m.$$

Matter perturbations in radiation dominated era

- Next, let us discuss the evolution of δ_m in the radiation dominated era.
- It turns out that at large scales $k\eta \to 0$, the potential ϕ does not evolve even in the radiation dominated era. The dark matter density perturbations also do not evolve outside the Hubble radius.
- At small scales, we can take the conservation equations for δ_m and $v_{m,\alpha}$ and put $\bar{P}_m = p_m = 0$:

$$\delta_m' = -\gamma^{\alpha\beta} \mathrm{i} k_\beta v_{m,\alpha} + 3\phi',$$

$$v_{m,\alpha}' = -4\frac{a'}{a} v_{m,\alpha} - \frac{\bar{\rho}_m'}{\bar{\rho}_m} v_{m,\alpha} - \mathrm{i} k_\alpha \phi = -\frac{a'}{a} v_{m,\alpha} - \mathrm{i} k_\alpha \phi,$$

  where we have used the fact that $\bar{\rho}_m'/\bar{\rho}_m = -3a'/a$.
- We now differentiate the first equation and use the second to get a equation for δ:

$$\delta_m'' + \frac{a'}{a}\delta_m' = 3\phi'' + 3\frac{a'}{a}\phi' - k^2\phi.$$

- For the radiation dominated era, $a'/a = 1/\eta$.
- It turns out that for small scales in the radiation dominated era, the growing solution is given by the homogeneous part

$$\delta_m = C_1(\vec{k}) + C_2(\vec{k}) \ln \eta.$$

  Thus, at best, the perturbations can grow logarithmically, which is very slow.
- Hence, to summarize, in the radiation dominated era δ_m does not grow in any scale (ignoring the slow logarithmic growth at small scales). In the matter dominated era, δ_m can grow only at scales smaller than the Hubble radius.

Evolution summary

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	radiation-dominated	matter-dominated
large scales ($k\eta \ll 1$)	ϕ constant, δ_m constant	ϕ constant, δ_m constant
small scales ($k\eta \gg 1$)	ϕ oscillates, decreases, $\delta_m \propto \ln \eta$ (almost constant)	ϕ constant, $\delta_m \propto a$

Transfer function

- Let us assume that the fluctuations originate during the inflationary epoch. We shall take the end of inflation a_{end} to be our $a = 0$.
- The evolution of the potential to the present epoch can be characterised by the **transfer function**

$$\phi(\eta_0, k) = \phi(a = 1, k) \equiv \phi(a = 0, k) T_1(k),$$

  where $\phi(k, a = 0)$ is the primordial potential.
- The transfer function is normalized such that it is unity at very large scales ($k \to 0$)

$$T(k) \equiv \frac{T_1(k)}{T_1(0)} \quad \Longrightarrow \quad T(k) = \frac{\phi(a = 1, k)}{\phi(a = 0, k)} \times \frac{\phi(a = 0, 0)}{\phi(a = 1, 0)}.$$

- The non-trivial form of $T(k)$ will arise because ϕ evolves differently in the radiation-dominated and matter-dominated era.
- So we have

$$\phi(a = 1, k) = \phi(a = 0, k) T(k) \left[\frac{\phi(a = 1, 0)}{\phi(a = 0, 0)}\right].$$

- We can write the power spectrum for potential as (ignoring a volume normalization factor)

$$P_\phi(a, k) = |\phi(a, k)|^2.$$

Matter power spectrum

- For scales $\ll H^{-1}$, the matter density contrast is related to the potential through the usual Poisson equation

$$\delta_m = -\frac{k^2\phi}{4\pi G a^2 \bar{\rho}} = -\frac{k^2\phi a}{4\pi G\bar{\rho}_0} = -\frac{2k^2\phi a}{3H_0^2\Omega_{m,0}},$$

- At the present epoch, for scales $\ll H_0^{-1} \approx 3000h^{-1}$ Mpc, we have

$$\delta_m(a=1,k) = -\frac{2k^2\phi(a=1,k)}{3H_0^2\Omega_{m,0}} = -\frac{2}{3H_0^2\Omega_{m,0}} k^2\phi(a=0,k)T(k)\left[\frac{\phi(a=1,0)}{\phi(a=0,0)}\right].$$

- The density power spectrum of matter fluctuations is defined as (apart from a volume normalization)

$$P_m(a,k) = |\delta_m(a,k)|^2 .$$

- The density power spectrum at small scales at the present epoch is related to the primordial potential power spectrum as (retaining only the k-dependent part) $P_m(a=1,k) \propto k^4 P_\phi(a=0,k)T^2(k)$.
- Usually, $P_\phi(a=0,k)$ is assumed (or often predicted by inflationary models) to be of the scale-free power-law form

$$k^3 P_\phi(a=0,k) \propto k^{n-1}.$$

  For scale-invariant spectrum, we have $n=1$.
- The linearly extrapolated density power spectrum at the present epoch is then

$$P_m(a=1,k) = A_s\, k^n\, T^2(k).$$

  The normalization A_s is fixed by comparing with observations.

Initial conditions

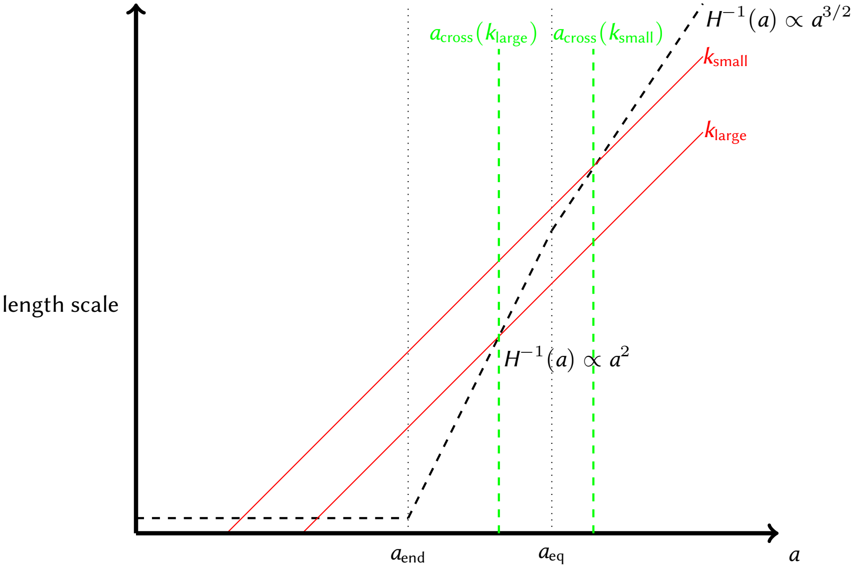
- Once inflation ends, all the length of interest are larger than the Hubble radius $H^{-1}(a)$.
- Inflationary models would predict the value of $\phi(a_{\rm end}, k) \equiv \phi(0, k)$. The corresponding power spectrum is the primordial power spectrum.
- For a mode larger than the Hubble radius, the potential does not grow. Let $a_{\rm cross}$ denote the time when a mode enters the Hubble radius ($a_{\rm cross}$ is hence a function of k).
- Then we can write

$$\phi(a_{\rm cross}, k) = \phi(0, k).$$

- For scales larger than Hubble radius, the density contrast too does not grow and is given by $\delta_m \propto -\phi$, which we write as $\delta_m = -C\phi$, where C is a constant (which can be shown to be 2, but not important for this analysis).
- Thus we can write

$$\delta_m(a_{\rm cross}, k) = -C\phi(0, k).$$

Crossing the Hubble radius



Dependence of crossing time on scales

- Next let us estimate the dependence of scales k^{-1} on their time $a_{\rm cross}$ of crossing the Hubble radius $H^{-1}(a)$.
- The proper scale evolves as a/k. The Hubble radius $H^{-1}(a) \propto a^{3/2}$ in the matter-dominated epoch and $H^{-1}(a) \propto a^2$ in the radiation-dominated epoch.
- Suppose there are two scales k_1 and k_2 which enter the Hubble radius in the radiation dominated epoch. Then their crossing times will be related by

$$\frac{a_{\rm cross}(k_1)}{a_{\rm cross}(k_2)} = \frac{k_2}{k_1} \qquad \text{(RD)}.$$

- For similar two scales in the matter dominated epoch, the relation is

$$\frac{a_{\rm cross}(k_1)}{a_{\rm cross}(k_2)} = \frac{k_2^2}{k_1^2} \qquad \text{(MD)}.$$

- If $k_{\rm eq}$ denotes the scale which crosses the Hubble radius at $a = a_{\rm eq}$, then scales $k \gg k_{\rm eq}$ cross during the radiation-dominated epoch and have $a_{\rm cross} \ll a_{\rm eq}$.
- Similarly, scales $k \ll k_{\rm eq}$ cross during the matter-dominated epoch and have $a_{\rm cross} \gg a_{\rm eq}$.
- Hence, we can write (approximately)

$$\frac{k}{k_{\rm eq}} = \left(\frac{a_{\rm eq}}{a_{\rm cross}}\right)^{1/2} \qquad \text{for } a_{\rm cross} \gg a_{\rm eq} \text{ (MD)},$$

$$\frac{k}{k_{\rm eq}} = \left(\frac{a_{\rm eq}}{a_{\rm cross}}\right) \qquad \text{for } a_{\rm cross} \ll a_{\rm eq} \text{ (RD)}.$$

Density evolution for different scales

- Now, consider the density contrasts which cross the Hubble radius during the radiation-dominated epoch, i.e., those corresponding to scales $k \gg k_{\rm eq}$. Till $a = a_{\rm eq}$ these modes grow only logarithmically (which we will ignore here). So $\delta_m(a_{\rm eq}, k) \approx \delta_m(a_{\rm cross}, k)$.
- Once they enter the matter dominated epoch, they grow $\propto a$. Hence

$$\delta_m(a, k) = \delta_m(a_{\rm eq}, k) \left(\frac{a}{a_{\rm eq}}\right) = \delta_m(a_{\rm cross}, k) \left(\frac{a}{a_{\rm eq}}\right) \qquad \text{for } k \gg k_{\rm eq} \text{ (i.e., } a_{\rm cross} \ll a_{\rm eq}\text{)}.$$

- Modes which cross during the matter-dominated epoch, on the other hand, follow the relation

$$\delta_m(a, k) = \delta_m(a_{\rm cross}, k) \left(\frac{a}{a_{\rm cross}}\right) \qquad \text{for } k \ll k_{\rm eq} \text{ (i.e., } a_{\rm cross} \gg a_{\rm eq}\text{)}.$$

- For $a_{\rm cross} \gg a_{\rm eq}$ (modes crossing in matter-dominated), we write $a/a_{\rm cross} = (a/a_{\rm eq}) \times (a_{\rm eq}/a_{\rm cross}) = (k^2/k_{\rm eq}^2)(a/a_{\rm eq})$, so

$$\delta_m(a, k) = \delta(a_{\rm cross}, k) \left(\frac{k}{k_{\rm eq}}\right)^2 \left(\frac{a}{a_{\rm eq}}\right) \qquad \text{for } k \ll k_{\rm eq} \text{ (i.e., } a_{\rm cross} \gg a_{\rm eq}\text{)}.$$

- We can now relate the density contrast to the primordial power spectrum as

$$\delta_m(a, k) = -C\phi(0, k) \left(\frac{a}{a_{\rm eq}}\right) \qquad \text{for } k \gg k_{\rm eq},$$

$$\delta_m(a, k) = -C\phi(0, k) \left(\frac{k}{k_{\rm eq}}\right)^2 \left(\frac{a}{a_{\rm eq}}\right) \qquad \text{for } k \ll k_{\rm eq}.$$

Simple estimates for $T(k)$

- We have seen that (for scales smaller than the Hubble radius at the present epoch)

$$\delta_m(a=1,k) \propto k^2 \phi(a=0,k) T(k).$$

- Hence, we have

$$T(k) \propto k^{-2} a_{\rm eq}^{-1} \propto \frac{k_{\rm eq}^2}{k^2} \qquad \text{for } k \gg k_{\rm eq},$$
$$T(k) \propto \text{const} = 1 \qquad \text{for } k \ll k_{\rm eq}.$$

  This implies that the transfer function has a "feature" around $k = k_{\rm eq}$.

- If the primordial power spectrum is of a power-law form, then

$$P_m(a=1,k) \propto k^{n-4} \qquad \text{for } k \gg k_{\rm eq},$$
$$P_m(a=1,k) \propto k^{n} \qquad \text{for } k \ll k_{\rm eq}.$$

  For $n \approx 1$, we have $P_m(a=1,k) \propto k^{-3}$ at small scales and $\propto k$ at large scales. The turnover happens around $k = k_{\rm eq}$.

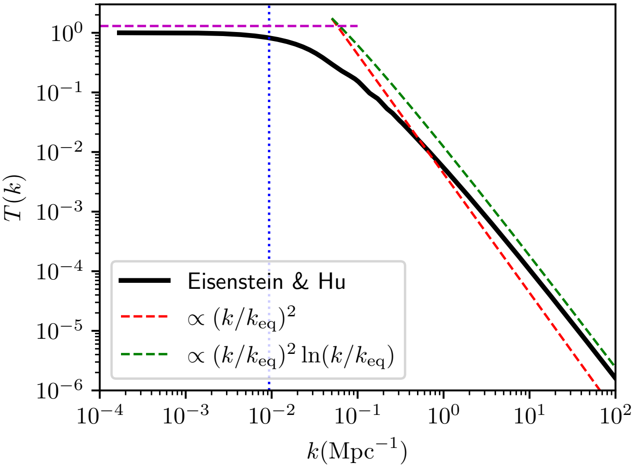
- The value of $k_{\rm eq}$ is determined by the condition

$$\frac{a_{\rm eq}}{k_{\rm eq}} = \frac{1}{H(a_{\rm eq})} \quad \Longrightarrow \quad k_{\rm eq} \approx 0.07\, \Omega_{m,0} h^2 \text{Mpc}^{-1}.$$

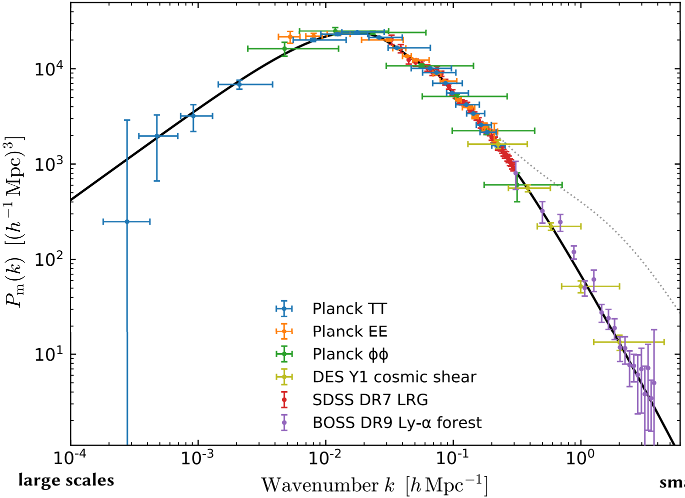
  Observations of this turnover then can put constraints on the value of $\Omega_{m,0} h^2$.

Numerical results

- There are fitting functions available for $T(k)$. The most used ones are by Eisenstein & Hu and BBKS.
- Also, codes like CAMB & CLASS calculate $T(k)$ by solving the full set of equations and provide $T(k)$ as a table.



The matter power spectrum



large scales **small scales**

$P_m(k)$ contains information on the primordial fluctuations and subsequent evolution

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