Cosmology
Lecture 11
Relativistic perturbation theory

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Perturbations

- Till now, we have treated the universe to be perfectly homogeneous / smooth which is a good assumption when averaged over large scales.
- However, to explain the structures around us (galaxies, clusters etc), we need to introduce inhomogeneities as perturbations on top of the smooth background.
- The basic idea is to use a metric $g_{ij} = \bar{g}_{ij} + \delta g_{ij}$ (which will lead to a Einstein tensor $G_{ij} = \bar{G}_{ij} + \delta G_{ij}$) and a source tensor $T_{ij} = \bar{T}_{ij} + \delta T_{ij}$. The unperturbed part $\bar{G}_{ij} = 8\pi G\, \bar{T}_{ij}$ would lead to the Friedmann equations for the smooth universe.
- The perturbed part $\delta G_{ij} = 8\pi G\, \delta T_{ij}$ can used for studying the evolution of the perturbations.
- If the amplitude of these perturbations is small, we can use linear perturbation theory and work out the mathematics relatively easily.

The perturbed metric

- The background metric is given by (assuming spatially flat)

$$ds^2 = dt^2 - a^2(t)\gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta, \quad \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta = dr^2 + r^2 d\Omega^2.$$

  In terms of Cartesian coordinates, $\gamma_{\alpha\beta} = \delta_{\alpha\beta}$.
- Recall: x^α is the *comoving coordinate* of a cosmological observer, while the *proper coordinate* would be $a(t)x^\alpha$.
- It is more convenient to use the **conformal time coordinate** $d\eta = dt/a(t)$, in terms of which the metric becomes

$$ds^2 = a^2(\eta)\ \left(d\eta^2 - \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta\right).$$

- Note that the Hubble parameter is given by $H(\eta) = \dot{a}/a = a'/a^2$.
- Now, when the universe is perturbed, the metric takes the form

$$ds^2 = (\bar{g}_{ij} + \delta g_{ij})\, dx^i dx^j = a^2(\eta)\ \left[(1 + 2\psi)\, d\eta^2 - 2w_\alpha\, d\eta\, dx^\alpha - (\gamma_{\alpha\beta} + h_{\alpha\beta})\, dx^\alpha\, dx^\beta\right].$$

- Note that δg_{ij} has ten independent components, which have been written as a scalar $\psi(\eta, \vec{x})$, a three-vector $w_\alpha(\eta, \vec{x})$ and a symmetric second-rank tensor $h_{\alpha\beta}(\eta, \vec{x})$. These characterizations are with respect to rotations in the three-space.
- We will assume that the perturbations are small, i.e., $\psi \ll 1,\ |w_\alpha| \ll 1, |h_{\alpha\beta}| \ll |\gamma_{\alpha\beta}|$. These perturbations must vanish when averaged over large scales.

Decomposition of the perturbations

- One can decompose a three-vector into a divergence-free and a curl-free component:

$$\vec{w} = \vec{w}^{\perp} + \vec{w}^{\parallel}, \quad \vec{\nabla} \cdot \vec{w}^{\perp} = \vec{\nabla} \times \vec{w}^{\parallel} = 0 \quad \Longrightarrow \quad \vec{w} = \vec{w}^{\perp} + \vec{\nabla} W, \quad \vec{\nabla} \cdot \vec{w}^{\perp} = 0.$$

- The corresponding decomposition could also be written as

$$w_\alpha = w_\alpha^{\perp} + \frac{\partial W}{\partial x^\alpha}, \quad \gamma^{\alpha\beta} \frac{\partial w_\alpha^{\perp}}{\partial x^\beta} = 0.$$

  Thus the three components of the vector w_α can be represented by a divergence-free vector $w_\alpha^{\perp}$ having two independent components and a scalar W.
- Similarly, we can expect that a tensor $h_{\alpha\beta}$ too can be decomposed too into tensor(s), vector(s) and scalar(s), with the tensor and vector being divergence-free.
- First, let us denote the trace of $h_{\alpha\beta}$ by -6ϕ. Then, we can define a traceless tensor as

$$s_{\alpha\beta} = \frac{1}{2}\left(h_{\alpha\beta} + 2\phi\gamma_{\alpha\beta}\right), \quad \gamma^{\alpha\beta} s_{\alpha\beta} = 0.$$

- One can further decompose the traceless $s_{\alpha\beta}$ into a scalar, a vector and a tensor as follows:

$$s_{\alpha\beta} = \left(\frac{\partial^2 S}{\partial x^\alpha \partial x^\beta} - \frac{1}{3}\gamma_{\alpha\beta}\gamma^{\mu\nu}\frac{\partial^2 S}{\partial x^\mu \partial x^\nu}\right) + \left(\frac{\partial s_\alpha^{\perp}}{\partial x^\beta} + \frac{\partial s_\beta^{\perp}}{\partial x^\alpha}\right) + s_{\alpha\beta}^{T}, \;\; \gamma^{\alpha\beta}\frac{\partial s_\alpha^{\perp}}{\partial x^\beta} = \gamma^{\mu\beta}\frac{\partial s_{\alpha\beta}^{T}}{\mathrm{d}x^\mu} = 0, \;\; \gamma^{\alpha\beta} s_{\alpha\beta}^{T} = 0.$$

- So, the perturbations are

$$(\phi, \psi, W, S) \longrightarrow 4 \text{ scalars}, \quad (w_\alpha^{\perp}, s_\alpha^{\perp}) \longrightarrow 2 \text{ vectors}, 2 \times 2 \text{ ind. comp.}, \quad (s_{\alpha\beta}^{T}) \rightarrow 1 \text{ tensor}, 2 \text{ ind. comp.}$$

Perturbed stress energy tensor

- The tensor for a perfect fluid is given by

$$T^i{}_k = (\rho + P)\, u^i u_k - P\,\delta^i{}_k,$$

where u^i is the four velocity with $u^i u_i = 1$.

- In the fluid rest frame $u^0 = 1$, $u^\alpha = 0$, hence the unperturbed tensor will be given by

$$\bar{T}^0{}_0 = \bar{\rho},\ \bar{T}^0{}_\alpha = 0 = \bar{T}^\alpha{}_0,\ \bar{T}^\alpha{}_\beta = -\bar{P}\,\delta^\alpha{}_\beta,$$

where $\bar{\rho}$ and $\bar{P}$ represent the unperturbed quantities.

- In the perturbed metric, the velocity u^k can be calculated by using $u_i u^i = 1$ and the components are given by

$$u^0 = \frac{1}{a}(1 - \psi),\ \ u_0 = a(1 + \psi),\ \ u^\alpha = \frac{1}{a} v^\alpha,\ \ u_\alpha = -a(v_\alpha + w_\alpha),$$

where $\vec{v}$ is the 3-velocity of the fluid. We have assumed $|\vec{v}| \ll 1$.

- The components of T^i_k are then

$$T^0{}_0 = \rho, \quad T^\alpha{}_0 = (\rho + P) v^\alpha, \quad T^0{}_\alpha = -(\rho + P)(v_\alpha + w_\alpha), \quad T^\alpha{}_\beta = -P(\delta^\alpha{}_\beta + \Sigma^\alpha{}_\beta).$$

- The off-diagonal terms of T^α_β (i.e, $\Sigma^\alpha{}_\beta$) vanish for a perfect fluid. These terms represent the anisotropic stress which arise for dissipative effects, e.g., viscosity, radiation-matter interaction, neutrinos. We shall ignore these effects for these lectures.

Density contrast and pressure perturbations

- Now, we assume that the density and pressure can be decomposed into an unperturbed part and a perturbations:

$$\delta(\eta, \vec{x}) = \frac{\rho(\eta, \vec{x}) - \bar{\rho}(\eta)}{\bar{\rho}(\eta)} = \frac{\rho(\eta, \vec{x})}{\bar{\rho}(\eta)} - 1, \quad p(\eta, \vec{x}) = P(\eta, \vec{x}) - \bar{P}(\eta),$$

  where δ is called the **density contrast** and p is the **perturbed pressure**.
- We will assume $\delta \ll 1, \;\; p \ll \bar{P}$ (or of the same order as δ if $\bar{P} = 0$), then the components of $T^i{}_k$ are (to the first order in perturbations)

$$T^0{}_0 = \bar{\rho}(1 + \delta), \quad T^\alpha{}_0 = (\bar{\rho} + \bar{P}) v^\alpha, \quad T^0{}_\alpha = -(\bar{\rho} + \bar{P})(v_\alpha + w_\alpha), \quad T^\alpha{}_\beta = -(\bar{P} + p)\delta^\alpha{}_\beta.$$

Decomposition theorem

- The next step would be to use the above definitions of g_{ik} and T_{ik} in the Einstein equation $\delta G_{ik} = 8\pi G\, \delta T_{ik}$. In addition, one can also work with the conservation equations $\delta T^i_{k;i} = 0$, keeping in mind that they are already contained in the Einstein equations and hence are not independent.
- It can be shown that the scalar, vector and tensor parts do not couple to each other (in the first-order perturbation theory), but they evolve independently. This is known as the **decomposition theorem**.
- This allows us to treat them separately. We can study, e.g., scalar perturbations as if the vector and tensor perturbations were absent. The total evolution of the full perturbation is just a linear superposition of the independent evolution of the scalar, vector, and tensor part of the perturbation.
- In most cases of interest, we will consider only scalar perturbations, i.e., $w^{\perp}_{\alpha} = s^{\perp}_{\alpha} = s^{T}_{\alpha\beta} = 0$.
- The tensor perturbations correspond to gravitational radiation which we will not study at present. In particular, the two components of $s^{T}_{\alpha\beta}$ are simply the two polarizations of the gravitational wave.
- The vector perturbations correspond to gravitomagnetism, which too can be ignored. These perturbations couple to rotational velocity perturbations in the cosmic fluid. They tend to decay in an expanding universe, and are therefore probably not important in cosmology.
- The scalar perturbations are sourced by density and pressure perturbations which will be of interest to us.

Gauge transformations

- There is an additional complication in general relativity: the freedom in choosing the coordinate systems.
- For a homogeneous background, one conveniently chooses coordinates that represent the symmetry of the system (i.e., geodesic observers expanding with the universe, called "cosmological observers"). There is no such obvious coordinate system for analysing the perturbed universe.
- It is possible to choose different coordinate systems, differing by factor having amplitudes of the same order of the perturbations, which are all equally valid.
- A particular choice of these coordinates is called a **gauge**. The transformation between different gauges is called a **gauge transformation**.
- Now, given 10 components of the perturbation, one can make a coordinate transformation making 4 of the components vanish (or putting 4 constraints), thus giving us only 6 independent components. Choosing the gauge is equivalent to choosing a convenient coordinate system which makes calculations easier.
- A convenient gauge is the **Poisson gauge**, which is equivalent to constraints

$$W = S = s_\alpha^\perp = 0.$$

- It is also possible to define *gauge-invariant variables* which remain unchanged under infinitesimal coordinate transformations. For Poisson gauge, these variables turn out to be ψ, ϕ, $w_\alpha^\perp$.
- In Poisson gauge (which is also known as **Newtonian gauge** for scalar perturbations), the metric is given by

$$\mathrm{d}s^2 = a^2(\eta)\left[(1+2\psi)\mathrm{d}\eta^2 - (1-2\phi)\gamma_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta\right].$$

- The other popular gauge that is often used is the **synchronous gauge** where one chooses $\psi = W = w_\alpha^\perp = 0$.

Fourier transforms

- For the first order perturbations, the dynamical equations are linear in the perturbed quantities. It is often easier to work in the Fourier space where derivatives become simple multiplications.
- Assuming the spatial metric to be flat, for any quantity $f(\eta, \vec{x})$, we define the Fourier transform as

$$f(\eta, \vec{k}) = \int d^3x\, f(\eta, \vec{x})\, e^{-i\vec{k}\cdot\vec{x}}. = \int d^3x\, f(\eta, \vec{x})\, e^{-ik_\mu x^\mu}.$$

- The inverse relations are of the form (e.g., for one component of the velocity)

$$v_\alpha(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, v_\alpha(\eta, \vec{k})\, e^{ik_\mu x^\mu}.$$

- Note that

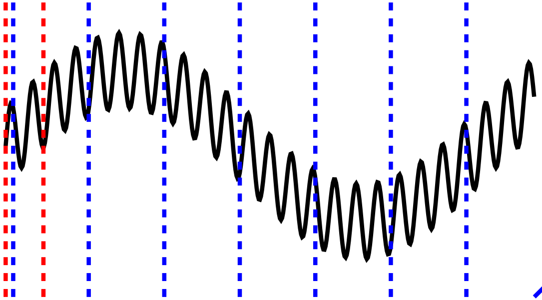
$$\frac{\partial v_\alpha(\vec{x})}{\partial x^\beta} = \int \frac{d^3k}{(2\pi)^3}\, v_\alpha(\vec{k}, \eta)\, e^{ik_\mu x^\mu}\, (ik_\mu \delta^\mu_\beta) = \int \frac{d^3k}{(2\pi)^3}\, \left[ik_\beta v_\alpha(\vec{k}, \eta)\right]\, e^{ik_\mu x^\mu},$$

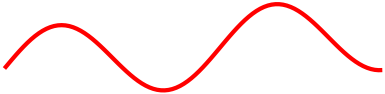
  implying that spatial derivatives $\partial/\partial x^\beta$ introduce factors of ik_β in Fourier space.

Fourier space: a simple toy example

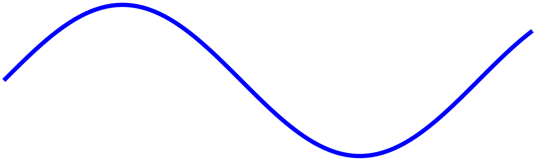
small scales: $\sin 22x$ (only $k = 22$)

$$f(x) = \sin x + \sin 22x \Longrightarrow f(k) \propto \delta_D(k-1) + \delta_D(k-22)$$





large scales: $\sin x$ (only $k = 1$)



Einstein equation

- It can be shown that the ${}^{\alpha}{}_{\beta\neq\alpha}$ component of the Einstein equations imply (when there is no anisotropic stress)

$$-\frac{1}{a^2}\gamma^{\alpha\mu}\frac{\partial^2(\phi-\psi)}{\partial x^{\mu}\partial x^{\beta}}=0 \quad (\alpha\neq\beta) \quad \Longrightarrow \quad \phi-\psi=0.$$

  This effectively reduces the number of perturbed quantities to one, which is the relativistic equivalent of the gravitational potential.
- The ${}^{0}{}_{0}$ component if the Einstein equations in Fourier space gives

$$3\frac{a'}{a}\phi'+\left(k^2+3\frac{a'^2}{a^2}\right)\phi=-4\pi G a^2\bar{\rho}\delta,$$

  where we have assumed that $\psi=\phi$.
- This is the relativistic version of **Poisson equation**. In the Newtonian limit, we have the equation as

$$\vec{\nabla}^2\phi=4\pi G a^2\bar{\rho}\delta \quad \Longrightarrow \quad k^2\phi=-4\pi G a^2\bar{\rho}\delta.$$

- Alternatively, one can us the ${}^{\alpha}{}_{\alpha}$ component which is

$$\phi''+3\frac{a'}{a}\phi'+2\frac{a''}{a}\phi-\frac{a'^2}{a^2}\phi=4\pi G a^2 p.$$

Conservation equations

- We can also use the conservation equations (keeping in mind that they are already contained in the Einstein equation)

$$\delta' = -\left(\gamma^{\alpha\beta}\frac{\partial v_\alpha}{\partial x^\beta} - 3\phi'\right)\left(1 + \frac{\bar{P}}{\bar{\rho}}\right) - 3\frac{a'}{a}\,\frac{p - \bar{P}\delta}{\bar{\rho}},$$

$$v'_\alpha = -4\frac{a'}{a}v_\alpha - \frac{\bar{\rho}' + \bar{P}'}{\bar{\rho} + \bar{P}}v_\alpha - \frac{1}{\bar{\rho} + \bar{P}}\frac{\partial p}{\partial x^\alpha} - \frac{\partial \phi}{\partial x^\alpha},$$

- The first equation is the *continuity equation*, while the second is the *Euler equation*.
- In Fourier space

$$\delta' = -\left(\gamma^{\alpha\beta}\mathrm{i}k_\beta v_\alpha - 3\phi'\right)\left(1 + \frac{\bar{P}}{\bar{\rho}}\right) - 3\frac{a'}{a}\,\frac{p - \bar{P}\delta}{\bar{\rho}},$$

$$v'_\alpha = -4\frac{a'}{a}v_\alpha - \frac{\bar{\rho}' + \bar{P}'}{\bar{\rho} + \bar{P}}v_\alpha - \frac{1}{\bar{\rho} + \bar{P}}\mathrm{i}k_\alpha p - \mathrm{i}k_\alpha\phi.$$

- We can write the equations explicitly in terms of only the scalars by using $v_\alpha = \partial V/\partial x^\alpha$ which in Fourier space becomes $v_\alpha = \mathrm{i}k_\alpha V$. Then

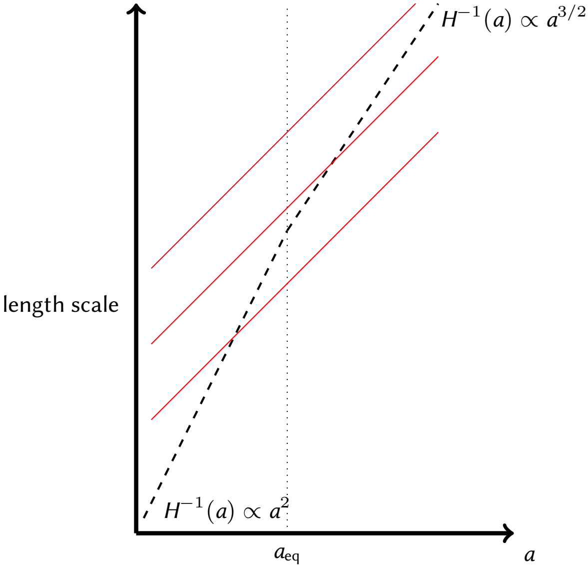
$$\delta' = \left(\gamma^{\alpha\beta}k_\alpha k_\beta V + 3\phi'\right)\left(1 + \frac{\bar{P}}{\bar{\rho}}\right) - 3\frac{a'}{a}\,\frac{p - \bar{P}\delta}{\bar{\rho}},$$

$$V' = -4\frac{a'}{a}V - \frac{\bar{\rho}' + \bar{P}'}{\bar{\rho} + \bar{P}}V - \frac{1}{\bar{\rho} + \bar{P}}p - \phi.$$

Hubble radius

- In general, we will study the perturbations in two extreme scales with respect to the **Hubble radius** $H^{-1}(a) = a^2/a'$.
- If we assume
$$a \propto t^n \implies \eta \propto t^{1-n} \implies a \propto \eta^{n/(1-n)} \implies a' \propto \eta^{(2n-1)/(1-n)} \implies a'/a \propto 1/\eta.$$
Thus $H^{-1} \propto a\eta$.
- Recall that k is conjugate to the comoving scale x, hence we should compare it with the **comoving Hubble radius** which is simply $H^{-1}/a \propto \eta$.
- Hence large scales would correspond to $k\eta \ll 1$, and small scales $k\eta \gg 1$.
- The comoving length scale corresponding to k is $2\pi/k$ and the physical scale is $\lambda = 2\pi a/k$. This should be compared with the Hubble radius $H^{-1} \propto a\eta$.
- At early times, η is small and hence most k values of interest are smaller than η^{-1}, i.e., $k\eta \ll 1 \implies \lambda \gg \eta$.
- As the universe expands, a physical length scale corresponding to k evolves as $\lambda \propto a$, while $H^{-1} \propto a\eta \propto a^{1/n}$. For $n = 1/2$ (radiation dominated), we get $H^{-1} \propto a^2$, while for $n = 2/3$ (matter dominated), we get $H^{-1} \propto a^{3/2}$.
- Thus, with time, the length scale λ will become smaller than H^{-1}. This is known as modes *entering the Hubble radius*.

Entering the Hubble radius



Entering the Hubble radius: impact of inflation