Cosmology
Lecture 8

Thermal history: equilibrium and decoupling

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Thermal equilibrium

- We have already seen that the momentum $|\vec{p}| \propto a^{-1}$, hence at early epochs, all particles will be ultra-relativistic (almost massless). More massive particles would become non-relativistic earlier.
- The temperature of the relativistic species $T \propto a^{-1}$, so the universe would be hotter at early times (hence the name *hot Big Bang*).
- If the particles interact among themselves, they can establish **thermal equilibrium**.
- Thermal equilibrium is established provided
  1. there is efficient energy and momentum exchange between the particles during scattering **(kinetic equilibrium)**,
  2. there is efficient creation and destruction of particles during the interaction **(chemical equilibrium)**.
- For example, when the temperature of radiation was higher than electron mass $k_B T > m_e \approx 0.511$ MeV, we expect electrons and positrons to be produced by pair production and they annihilate to produce photons $\gamma \longleftrightarrow e^- + e^+$.
- Photons would couple to (non-relativistic) $e^\pm$ via Thomson scattering as well $e^\pm + \gamma \longleftrightarrow e^\pm + \gamma$.
- These interactions would establish equilibrium between γ, e^+, e^-.
- There could also be (neutral current) weak interactions (mediated by Z-bosons) of the type $e^- + e^+ \longleftrightarrow \nu_e + \bar{\nu}_e$.
- Consequently, these interactions could establish a thermal equilibrium among these particles. The universe will be filled with a hot plasma consisting of $\gamma, e^-, e^+, \nu_e, \bar{\nu}_e$ all in thermal equilibrium at the same temperature.
- Such arguments can be extended to all particles in the standard model of particle physics. If some particles do not interact with others, they may not be in thermal equilibrium.

Detailed balance

- In absence of any interactions, the number density of particles of a species A evolves as $n_A \propto a^{-3}$ (effect of volume dilution), which can be written as a conservation equation

$$\dot{n}_A \equiv \frac{\mathrm{d}n_A}{\mathrm{d}t} = -\frac{3\dot{a}}{a} n_A = -3Hn_A.$$

  The presence of interactions would introduce additional terms on the right hand side.
- Let us for the moment limit ourselves to processes of the form $A + B \longleftrightarrow L + M$, which represent two-particle scattering or annihilation.
- The evolution of species A is given by

$$\dot{n}_A = -3Hn_A - \alpha n_A n_B + \beta n_L n_M.$$

- The parameter $\alpha = \langle \sigma v \rangle$ is the *thermally averaged cross section* for the interaction between A and B. The parameter β is the same quantity for the reverse interaction.
- It can be related to α by demanding that in equilibrium $-\alpha n_A^{(0)} n_B^{(0)} + \beta n_L^{(0)} n_M^{(0)} = 0$, where the densities with superscript (0) denote the equilibrium values. Hence

$$\beta = \alpha \frac{n_A^{(0)} n_B^{(0)}}{n_L^{(0)} n_M^{(0)}}.$$

  This is known as the **principle of detailed balance** which is useful for relating forward and backward reaction rates.

Comoving densities

- Hence the evolution equation becomes

$$\frac{\dot{n}_A}{n_A} + 3H = -\langle \sigma v \rangle n_B \left[1 - \frac{n_A^{(0)} n_B^{(0)}}{n_L^{(0)} n_M^{(0)}} \frac{n_L n_M}{n_A n_B} \right].$$

- The *interaction rate* for A is simply

$$\Gamma_A \equiv \langle \sigma v \rangle n_B.$$

- Let us define the comoving number densities, e.g., $n_{0,A} = n_A a^3$, then

$$\frac{\dot{n}_A}{n_A} = \frac{\mathrm{d} \ln n_A}{\mathrm{d}t} = \frac{\mathrm{d} \ln n_{0,A}}{\mathrm{d}t} - 3 \frac{\mathrm{d} \ln a}{\mathrm{d}t} = \frac{\dot{n}_{0,A}}{n_{0,A}} - 3H,$$

  hence

$$\frac{\dot{n}_{0,A}}{n_{0,A}} = -\Gamma_A \left[1 - \frac{n_{0,A}^{(0)} n_{0,B}^{(0)}}{n_{0,L}^{(0)} n_{0,M}^{(0)}} \frac{n_{0,L} n_{0,M}}{n_{0,A} n_{0,B}} \right].$$

- We can write in terms of a as the independent variable

$$\frac{\mathrm{d} \ln n_{0,A}}{\mathrm{d}a} = \frac{\mathrm{d} \ln n_{0,A}}{\mathrm{d}t} \frac{1}{\dot{a}} = \frac{\dot{n}_{0,A}}{n_{0,A}} \frac{1}{a\,H} \quad \Longrightarrow \quad \frac{\dot{n}_{0,A}}{n_{0,A}} = H \frac{\mathrm{d} \ln n_{0,A}}{\mathrm{d} \ln a}$$

- So the evolution of $n_{0,A}$ is given by

$$\frac{\mathrm{d} \ln n_{0,A}}{\mathrm{d} \ln a} = -\frac{\Gamma_A}{H} \left[1 - \frac{n_{0,A}^{(0)} n_{0,B}^{(0)}}{n_{0,L}^{(0)} n_{0,M}^{(0)}} \frac{n_{0,L} n_{0,M}}{n_{0,A} n_{0,B}} \right].$$

Decoupling and freeze out

- $$\frac{d \ln n_{0,A}}{d \ln a} = -\frac{\Gamma_A}{H} \left[1 - \frac{n_{0,A}^{(0)} n_{0,B}^{(0)}}{n_{0,L}^{(0)} n_{0,M}^{(0)}} \frac{n_{0,L} n_{0,M}}{n_{0,A} n_{0,B}} \right].$$

  The number of interactions the particle species encounter over the age of the universe will be $\sim \Gamma_A/H$. So the evolution is determined by the number of interactions $\sim \Gamma_A/H$ and the departure from equilibrium.
- When $\Gamma_A \gg H$, the system will naturally lead towards equilibrium. For example, if $n_{0,A} < n_{0,A}^{(0)}$ (and for simplicity assume all the other species have their equilibrium value), then the right hand side will be positive. Thus $n_{0,A}$ will increase rapidly because the interaction term is large. In case $n_{0,A} > n_{0,A}^{(0)}$, the density will decrease rapidly. The system will thus rapidly relax to a state where the abundances have their equilibrium values.
- Since the density of particles decreases because of expansion, it is expected that Γ_A will decrease as the universe expands.
- Although H also decreases with time, the decrease in Γ_A is typically faster. We then expect the condition $\Gamma_A \gg H$ to change over to the condition $\Gamma_A \lesssim H$ as the Universe evolves. In such a situation, a reaction which was initially able to remain in equilibrium eventually falls out of equilibrium.
- At this point, the particle species A is said to have **decoupled** from the equilibrium.
- When $\Gamma_A \ll H$, the right hand side is negligible and the $n_{0,A} \to$ constant (hence $n_A \propto a^{-3}$), which is the equilibrium density at the epoch of decoupling. This is the **freeze-out** value of the species.

Major milestones of the hot Big Bang

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Event	t (approx)	z (approx)	T (approx)	Description
Inflation	10^{-34} s ?	$\rightarrow \infty$	10^{16} GeV ?	perturbations generated
Baryogenesis	?	?	?	explain the observed baryon density through some mechanism without assuming any initial asymmetry in matter and antimatter.
EW phase transition	10^{-11} s	10^{15}	100 GeV	electroweak force “breaks” into weak and electromagnetic
QCD phase transition	10^{-5} s	10^{12}	100 MeV	quarks & gluons bind into protons & neutrons
Dark matter freeze-out	?	?	?	dark matter particles decouple
Neutrino decoupling	1 s	10^{10}	1 MeV	neutrinos decouple
Electron-positron annihilation	10 s	10^{9}	0.5 MeV	$e^{\pm}$ annihilate into photons
Big Bang nucleosynthesis	3 m	10^{8}	100 keV	nuclei of light elements form
Matter-radiation equality	6×10^{4} y	3500	1 eV	
Recombination	4×10^{5} y	1100	0.2 eV	neutral hydrogen atoms form
Photon decoupling	4×10^{5} y	1100	0.2 eV	photons decouple from matter, CMB originates
Formation of first stars	10^{8} y	15	5 meV	first galaxies form
Dark energy-matter equality	10^{10} y	0.4	0.3 meV	
Present	1.4×10^{10} y	0	0.2 meV	

Equilibrium distribution function

- Since at early times, different species were in thermal equilibrium, we can describe the state using phase space distribution function (the number of particles per unit volume per unit momentum volume) $f_A(\vec{x}, \vec{p}, t)$.
- If the universe is homogeneous, then $f_A(\vec{x}, \vec{p}, t) \equiv f_A(\vec{p}, t)$. Further, isotropy implies $f_A(\vec{p}, t) \equiv f_A(p, t)$.
- For a species A in thermal equilibrium, the phase space distribution is given by

$$f_A(p, t) = \frac{g_A}{(2\pi\hbar)^3} \frac{1}{e^{[E(p)-\mu_A]/k_B T_A(t)} \pm 1},$$

  1. g_A is the spin-degeneracy factor for the species (it is 1 for neutrinos, 2 for photons and charged leptons and 6 for quarks),
  2. μ_A is the chemical potential,
  3. $E(p) = \sqrt{p^2 + m_A^2}$,
  4. $T_A(t)$ is the temperature characterizing the species and
  5. the upper sign corresponds to fermions and the lower one to bosons.
- Some important macroscopic/thermodynamic quantities are:
  - The number density: $n_A = \int d^3p\, f_A(p) = 4\pi \int_0^\infty dp\, p^2\, f_A(p) = \frac{g_A}{2\pi^2\hbar^3} \int_{m_A}^\infty \frac{dE\, E\sqrt{E^2 - m_A^2}}{e^{(E-\mu_A)/k_B T_A} \pm 1}$.
  - The energy density: $\rho_A = \int d^3p\, E\, f_A(p) = 4\pi \int_0^\infty dp\, p^2\, \sqrt{p^2 + m_A^2}\, f_A(p) = \frac{g_A}{2\pi^2\hbar^3} \int_{m_A}^\infty \frac{dE\, E^2\sqrt{E^2 - m_A^2}}{e^{(E-\mu_A)/k_B T_A} \pm 1}$.
  - The pressure is given by kinetic theory as $P = n\langle pv\rangle/3 = n\langle p\, p/E\rangle/3 = n\langle p^2/E\rangle/3$. Hence

$$P_A = \int d^3p\, \frac{p^2}{3E}\, f_A(p) = 4\pi \int_0^\infty dp\, \frac{p^4}{3\sqrt{p^2 + m_A^2}}\, f_A(p) = \frac{g_A}{6\pi^2\hbar^3} \int_{m_A}^\infty \frac{dE\, (E^2 - m_A^2)^{3/2}}{e^{(E-\mu_A)/k_B T_A} \pm 1}.$$

(Ultra-)relativistic limit

- In the relativistic limit $k_B T_A \gg m_A$. Also, in the early universe, $k_B T_A \gg \mu_A$ (we will argue later).
- Then

$$n_A = \frac{g_A}{2\pi^2 \hbar^3} \int_{m_A}^{\infty} \frac{\mathrm{d}E\, E \sqrt{E^2 - m_A^2}}{\mathrm{e}^{(E-\mu_A)/k_B T_A} \pm 1} \approx \frac{g_A}{2\pi^2 \hbar^3} \int_0^{\infty} \frac{\mathrm{d}E\, E^2}{\mathrm{e}^{E/k_B T_A} \pm 1} = \frac{g_A}{2\pi^2} \left(\frac{k_B}{\hbar}\right)^3 T_A^3 \int_0^{\infty} \frac{\mathrm{d}y\, y^2}{\mathrm{e}^y \pm 1}.$$

- For bosons, use the Riemann-zeta function $\zeta(n) = \frac{1}{\Gamma(n)} \int_0^{\infty} \frac{\mathrm{d}y\, y^{n-1}}{\mathrm{e}^y - 1}$.
- For fermions, use $\int_0^{\infty} \frac{\mathrm{d}y\, y^{n-1}}{\mathrm{e}^y + 1} = \int_0^{\infty} \frac{\mathrm{d}y\, y^{n-1}}{\mathrm{e}^y - 1} - 2 \int_0^{\infty} \frac{\mathrm{d}y\, y^{n-1}}{\mathrm{e}^{2y} - 1} = \left(1 - \frac{1}{2^{n-1}}\right) \Gamma(n)\zeta(n)$.
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$$\begin{aligned} n_A &= \frac{\zeta(3)}{\pi^2} \left(\frac{k_B}{\hbar}\right)^3 g_A T_A^3 && \text{for bosons,} \\ &= \frac{3}{4} \frac{\zeta(3)}{\pi^2} \left(\frac{k_B}{\hbar}\right)^3 g_A T_A^3 && \text{for fermions.} \end{aligned}$$

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$$\begin{aligned} \rho_A &= \frac{\pi^2}{30} \left(\frac{k_B^4}{\hbar^3}\right) g_A T_A^4 && \text{for bosons,} \\ &= \frac{7}{8} \frac{\pi^2}{30} \left(\frac{k_B^4}{\hbar^3}\right) g_A T_A^4 && \text{for fermions.} \end{aligned}$$

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$$P_A = \frac{1}{3} \rho_A \quad \text{for bosons and fermions.}$$

Non-relativistic limit

- For non-relativistic particles $k_B T_A \ll m_A$

$$n_A = \frac{g_A}{2\pi^2\hbar^3}\int_{m_A}^{\infty}\frac{\mathrm{d}E\,E\sqrt{E^2-m_A^2}}{\mathrm{e}^{(E-\mu_A)/k_BT_A}\pm 1} \approx \frac{g_A}{2\pi^2\hbar^3}\int_{m_A}^{\infty}\mathrm{d}E\,E\sqrt{E^2-m_A^2}\,\mathrm{e}^{-(E-\mu_A)/k_BT_A}$$

$$= \frac{g_A}{2\pi^2\hbar^3} m_A^3\,\mathrm{e}^{\mu_A/k_BT_A}\int_1^{\infty}\mathrm{d}x\,x\sqrt{x^2-1}\,\mathrm{e}^{-xm_A/k_BT_A} \approx g_A\left(\frac{m_Ak_B}{2\pi\hbar^2}\right)^{3/2} T_A^{3/2}\,\mathrm{e}^{(\mu_A-m_A)/k_BT_A}.$$

- The density is given by

$$\rho_A = n_A m_A + \frac{3}{2} n_A k_B T_A.$$

- The pressure is

$$P_A = n_A k_B T_A.$$

- Imagine two species 'NR' and 'R' at the same temperature T (e.g., non-relativistic electrons interacting with photons via Compton scattering), then note that

$$\frac{n_{\mathrm{NR}}}{n_R} = g_{\mathrm{NR}}\left(\frac{m_{\mathrm{NR}}k_BT}{2\pi\hbar^2}\right)^{3/2}\mathrm{e}^{(\mu_{\mathrm{NR}}-m_{\mathrm{NR}})/k_BT}\times\left(f_{B,F}\frac{\zeta(3)}{\pi^2}\frac{k_B^3}{\hbar^3}g_RT^3\right)^{-1} \sim \left(\frac{m_{\mathrm{NR}}}{k_BT}\right)^{3/2}\mathrm{e}^{-m_{\mathrm{NR}}/k_BT}.$$

- Note that we have $m_{\mathrm{NR}} \gg k_BT$. Thus the number (and energy) density of non-relativistic particles is exponentially damped by the factor $\mathrm{e}^{-m_{\mathrm{NR}}/k_BT}$ with respect to the relativistic particles.
- When a species becomes non-relativistic, the pair production cannot occur. However, the existing pairs can annihilate, thus reducing the number of particles.

Evolution after decoupling

- Consider a species A which decouples from the rest of the matter at $a = a_D$. Subsequently, the species would not interact with itself or any other species, hence the corresponding particles will have momenta $p \propto a^{-1}$.
- So the phase space distribution at $a > a_D$ would be $f_A(a, p) = f_A(a_D, pa/a_D)$.
- The distribution of a relativistic species (with $E, p \gg m$) at $a = a_D$ is $f_A(a_D, p) \approx \frac{g_A}{(2\pi\hbar)^3} \frac{1}{\mathrm{e}^{p/k_B T_D} \pm 1}$.
- Now, since they do not interact at $a > a_D$, we must have $f_A(a, p) = \frac{g_A}{(2\pi\hbar)^3} \frac{1}{\mathrm{e}^{pa/a_D k_B T_D} \pm 1}$.
- Thus the phase space distribution will retain the equilibrium form with T_D substituted by $T_A = a_D T_D / a$, as long as the particles remain relativistic, which ensures that T_A scale as a^{-1}.
- Note that if the particle A is non-relativistic at the time of decoupling ($E \approx m_A + p^2/2m_A$), assuming $\mu_A \ll k_B T_D$,

$$f_A(a_D, p) = \frac{g_A}{(2\pi\hbar)^3} \frac{1}{\mathrm{e}^{(m_A - \mu_A)/k_B T_D}\ \mathrm{e}^{p^2/(2 m_A k_B T_D)}} \approx \frac{g_A}{(2\pi\hbar)^3} \mathrm{e}^{-m_A/k_B T_D}\ \mathrm{e}^{-p^2/2 m_A k_B T_D}.$$

- The distribution at a later time is simply given by

$$f_A(a, p) = \frac{g_A}{(2\pi\hbar)^3} \mathrm{e}^{-m_A/k_B T_D}\ \mathrm{e}^{-p^2 a^2/2 m_A k_B T_D a_D^2},$$

  which has the Maxwell-Boltzmann form with temperature $T_A = T_D (a_D/a)^2$, thus $T_A \propto a^{-2}$.
- In case μ_A cannot be ignored, one can find the evolution of μ_A from entropy conservation (discussed later).
- In case the species is neither ultra-relativistic nor non-relativistic, the distribution function would not retain the equilibrium form in absence of interactions and cannot be described by a simple evolution of temperature.

Relativistic degrees of freedom

- In the early universe, the total ρ is dominated by the relativistic species (both thermal and decoupled).
- For the thermal part, all the species have the same temperature T, so we can write

$$\rho_{R,\mathrm{th}} = \sum_{B \in \mathrm{bosons}} \frac{\pi^2}{30} \left(\frac{k_B^4}{\hbar^3}\right) g_B T^4 + \sum_{F \in \mathrm{fermions}} \frac{7}{8} \frac{\pi^2}{30} \left(\frac{k_B^4}{\hbar^3}\right) g_F T^4 \equiv \frac{\pi^2}{30} \left(\frac{k_B^4}{\hbar^3}\right) g_{*,\mathrm{th}} T^4, \quad g_{*,\mathrm{th}} = \sum_{B \in \mathrm{bosons}} g_B + \frac{7}{8} \sum_{F \in \mathrm{fermions}} g_F.$$

- The decoupled species may have temperatures different from T, hence we write

$$\rho_{R,\mathrm{dec}} = \sum_{B \in \mathrm{bosons}} \frac{\pi^2}{30} \left(\frac{k_B^4}{\hbar^3}\right) g_B T_B^4 + \sum_{F \in \mathrm{fermions}} \frac{7}{8} \frac{\pi^2}{30} \left(\frac{k_B^4}{\hbar^3}\right) g_F T_F^4.$$

- In this case too, we write

$$\rho_{R,\mathrm{dec}} = \frac{\pi^2}{30} \left(\frac{k_B^4}{\hbar^3}\right) g_{*,\mathrm{dec}}(T) T^4, \quad g_{*,\mathrm{dec}}(T) = \sum_{B \in \mathrm{bosons}} g_B \left(\frac{T_B}{T}\right)^4 + \frac{7}{8} \sum_{F \in \mathrm{fermions}} g_F \left(\frac{T_F}{T}\right)^4.$$

- The *effective number of relativistic degrees of freedom* is defined as

$$g_*(T) = g_{*,\mathrm{th}} + g_{*,\mathrm{dec}}(T) \Longrightarrow \rho = \rho_R = \frac{\pi^2}{30} \left(\frac{k_B^4}{\hbar^3}\right) g_*(T) T^4.$$

- As T decreases, various species become non-relativistic and their contribution is removed from g_*, hence g_* decreases with time. For $k_B T \gg 175\mathrm{GeV}$, all standard model particles are relativistic (and in equilibrium), $g_* = 106.75$. At present, only photons and neutrinos (if massless) are relativistic, $g_* = 3.36$ (to be shown later).

Entropy of the universe

- Consider the formula for entropy from the second law of thermodynamics

$$T\,\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V = \mathrm{d}(\rho V) + P\mathrm{d}V = V\mathrm{d}\rho + (\rho + P)\mathrm{d}V.$$

- Treating S to be function of T, V, and $\rho \equiv \rho(T)$, $P \equiv P(T)$, we get the relations

$$\frac{\partial S}{\partial T} = \frac{V}{T}\frac{\mathrm{d}\rho}{\mathrm{d}T}, \quad \frac{\partial S}{\partial V} = \frac{\rho + P}{T}.$$

- Now use the integrability condition $\partial^2 S/\partial T\partial V = \partial^2 S/\partial V\partial T$ to obtain

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \frac{\rho + P}{T}.$$

- Inserting this in the second law, we get

$$T\mathrm{d}S = \mathrm{d}[(\rho + P)V] - V\mathrm{d}P = \mathrm{d}[(\rho + P)V] - (\rho + P)V\frac{\mathrm{d}T}{T} = T\mathrm{d}\left[\frac{(\rho + P)V}{T}\right].$$

- So upto an additive constant, the entropy can be defined as

$$S = \frac{(\rho + P)V}{T},$$

  which corresponds to a density

$$s = \frac{\rho + P}{T}.$$

Entropy of relativistic species

- For a species, the entropy density is

$$s_A = \frac{\rho_A + P_A}{T_A}.$$

  Clearly, the entropy for non-relativistic species will be exponentially suppressed.
- For a relativistic species

$$\begin{aligned} s_A = \frac{4\rho_A}{3T_A} &= \frac{2\pi^2}{45}\left(\frac{k_B^4}{\hbar^3}\right) g_A T_A^3 && \text{for bosons,} \\ &= \frac{7}{8}\frac{2\pi^2}{45}\left(\frac{k_B^4}{\hbar^3}\right) g_A T_A^3 && \text{for fermions.} \end{aligned}$$

- Similar to ρ, we write the total entropy density (essentially contributed by relativistic species) as

$$s = s_R = \frac{2\pi^2}{45}\left(\frac{k_B^4}{\hbar^3}\right) g_{*S}(T)\, T^3, \quad g_{*S}(T) = g_{*S,\text{th}} + g_{*S,\text{dec}}(T),$$

$$g_{*S,\text{th}} = \sum_{B\in\text{bosons}} g_B + \frac{7}{8}\sum_{F\in\text{fermions}} g_F = g_{*,\text{th}},$$

$$g_{*S,\text{dec}}(T) = \sum_{B\in\text{bosons}} g_B \left(\frac{T_B}{T}\right)^3 + \frac{7}{8}\sum_{F\in\text{fermions}} g_F \left(\frac{T_F}{T}\right)^3.$$

- The quantity $g_{*S}(T)$ is the *effective number of relativistic degrees of freedom in entropy.*

Conservation of entropy

- From Friedmann equations, we have

$$\mathrm{d}(\rho a^3) = -P\mathrm{d}(a^3) \Longrightarrow \mathrm{d}(\rho_R a^3) = -P_R \mathrm{d}(a^3),$$

  Note that the above may not hold for individual species if they are exchanging energy with others.
- Now, we also have

$$T\mathrm{d}S = \mathrm{d}(\rho V) + P\mathrm{d}V = \mathrm{d}(\rho a^3) + P\mathrm{d}(a^3) = 0.$$

  Hence the entropy $S = S_R = s_R a^3$ is conserved.
- One important consequence of the entropy conservation can be understood from the fact that $s_R\, a^3 = (2\pi^2/45)\,(k_B^4/\hbar^3)\, g_{*S}\, T^3 a^3$, and hence $T \propto a^{-1} g_{*S}^{-1/3}$.
- As long as g_{*S} remains constant, the temperature will decrease as $T \propto a^{-1}$ as expected.
- Whenever a species becomes non-relativistic, g_{*S} decreases, hence the combination $T \times a$ increases.
- In that case, conservation of entropy would imply that T will decrease slower than a^{-1} during the period when the species becomes non-relativistic. During this period, the entropy of the species becoming non-relativistic gets transferred to the rest of the relativistic particles. Once the species has become completely non-relativistic, it will no longer contribute to g_{*S}, and T will again keep on decreasing as a^{-1}.
- Thus T scales as a^{-1}, except that there is a change in the amplitude of the scaling every time g_{*S} changes.

Hubble expansion

- The expansion of the universe is given by (ignoring the curvature)

$$H^2(t) = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_R = \frac{8\pi G}{3} g_* \frac{\pi^2}{30}\left(\frac{k_B^4}{\hbar^3}\right) T^4 = \frac{4\pi^3}{45}\left(\frac{G k_B^4}{\hbar^3}\right) g_* T^4 \longrightarrow \frac{4\pi^3}{45}\left(\frac{G}{\hbar^3 c^5}\right) g_* (k_B T)^4$$

- As long as g_* does not change, we get $T \propto a^{-1}$ and we obtain the standard result $a \propto t^{1/2}$. In that case $H^2 = 1/4t^2$.
- In early radiation dominated epoch ($k_B T \gtrsim 100$ GeV), g_* is fairly constant, hence we get

$$t = \left(\frac{45}{16\pi^3}\right)^{1/2}\left(\frac{G}{\hbar^3 c^5}\right)^{-1/2} g_*^{-1/2} (k_B T)^{-2} \approx 2.42 \times 10^{-6} \text{ s } g_*^{-1/2}\left(\frac{k_B T}{\text{GeV}}\right)^{-2}.$$

- In general, we use $T \propto a^{-1} g_{*S}^{-1/3}$ to write $H^2 \propto \rho_R \propto \left(g_*/g_{*S}^{4/3}\right) a^{-4}$. This can be solved to obtain the expansion rate.

Baryon asymmetry

- Consider a relativistic fermion particle A and its antiparticle $\bar{A}$ in equilibrium with photons $A + \bar{A} \longleftrightarrow \gamma$ (pair production and annihilation).
- Since the photon number is not conserved in thermodynamic systems, we must have $\mu_\gamma = 0$. Then the above equilibrium state implies $\mu_{\bar{A}} = -\mu_A$.
- For the relativistic cases, we have

$$n_A = \frac{g_A}{2\pi^2\hbar^3} \int_{m_A}^{\infty} \frac{\mathrm{d}E\, E\sqrt{E^2 - m_A^2}}{\mathrm{e}^{(E-\mu_A)/k_B T_A} + 1} \approx \frac{g_A}{2\pi^2\hbar^3} \int_0^{\infty} \frac{\mathrm{d}E\, E^2}{\mathrm{e}^{(E-\mu_A)/k_B T_A} + 1}, \quad n_{\bar{A}} \approx \frac{g_A}{2\pi^2\hbar^3} \int_0^{\infty} \frac{\mathrm{d}E\, E^2}{\mathrm{e}^{(E+\mu_A)/k_B T_A} + 1}.$$

- One can show that (for $k_B T_A \gg m_A$)

$$n_A - n_{\bar{A}} \approx \frac{g_A}{6\pi^2\hbar^3} (k_B T_A)^3 \left[\pi^2 \left(\frac{\mu_A}{k_B T_A}\right) + \left(\frac{\mu_A}{k_B T_A}\right)^3\right].$$

  This should hold for species like protons and electrons.

Chemical potential

- Now, the baryons in the universe are mostly hydrogen, whose mass is contributed by protons. In that case, we can assume that the net number of baryons would be the difference between protons and anti-protons at the early epochs (when protons were relativistic). Hence

$$n_b = n_{p^+} - n_{p^-} \approx \frac{g_p}{6\pi^2\hbar^3}(k_B T)^3 \left[\pi^2 \left(\frac{\mu_p}{k_B T}\right) + \left(\frac{\mu_p}{k_B T}\right)^3\right].$$

- Now, we know that at the present epoch

$$n_{b,0} = \frac{\rho_b}{m_p} = \frac{\rho_{c,0}\Omega_{b,0}}{m_p} \approx 1.12 \times 10^{-5}\ \mathrm{cm}^{-3}\Omega_{b,0}h^2, \quad n_{\gamma,0} = \frac{\zeta(3)}{\pi^2}\left(\frac{k_B}{\hbar}\right)^3 2T_0^3 = 4.13 \times 10^2\ \mathrm{cm}^{-3}.$$

- Using $\Omega_{b,0}h^2 \approx 0.02$, we get $n_{b,0}/n_{\gamma,0} \sim 5 \times 10^{-10}$.
- This ratio should remain the same (at least to within order of magnitude) at early epochs as well.
- Now, if the protons, anti-protons and photons were in equilibrium at the same temperature T, then we have

$$\frac{n_b}{n_\gamma} \approx \frac{g_p}{12\zeta(3)}\left[\pi^2 \left(\frac{\mu_p}{k_B T}\right) + \left(\frac{\mu_p}{k_B T}\right)^3\right].$$

- Since this is a quantity $\sim 10^{-10}$, we expect $\mu_p/k_B T \sim 10^{-10}$. Thus, for all calculations, we can take $\mu_p \to 0$.
- The same argument holds for other particles, e.g., electrons as well. For example, we can use charge neutrality to argue that $n_{e^-} - n_{e^+} = n_{p^+} - n_{p^-} = n_b$, and the same conclusion can be reached for electrons.