Cosmology
Lecture 2
The FLRW metric

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Cosmological principle

- The most fundamental assumption in cosmology is that the Universe is homogeneous and isotropic when smoothed over very large scales ($\gtrsim 100$ Mpc). This is validated by observations of galaxies around us. On small scales, we see a lot of structure, however at larger scales, the statistical properties look the same at all points and along all directions.
- "Isotropy" is the claim that the universe looks the same in all direction. Direct evidence comes from the smoothness of the temperature of the cosmic microwave background.
- "Homogeneity" is the claim that the universe looks the same at every point. It is harder to test directly, although some evidence comes from number counts of galaxies.
- More traditionally, we may invoke the "Copernican principle" that we do not live in a special place in the universe. Then it follows that, since the universe appears isotropic around us, it should be isotropic around every point; and a basic theorem of geometry states that isotropy around every point implies homogeneity.
- We may therefore approximate the universe as a spatially homogeneous and isotropic while studying its large-scale properties. This is known as the **cosmological principle**.

Fundamental observers

- Note that at a given point, there is a preferred set of observers who see the Universe as isotropic. Such observers are known as **fundamental observers**, or **comoving observers**.
- For example, if an observer is moving along some direction with respect to a fundamental observer, they will observe galaxies moving towards them in the forward direction while they will move away in the backward direction. This will violate isotropy of the space.
- Similarly, an observer moving with respect to the fundamental observer will find a dipole anisotropy in the isotropic CMB radiation field.
- Such observers have what are known as "peculiar motions". These are usually because of local gravity (e.g., the nearby galaxies may be falling towards us).

Hubble's observations

- Hubble, in 1929, could measure distances to distant sources (which he called as nebulae, and we currently know them to be galaxies) using "Cepheid variables". He also had the measurements of the shift in the wavelength of spectral lines observed in these sources (which was done earlier by Slipher in 1914).
- The shift in the wavelength can be used to define a quantity called redshift z:

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} \equiv 1 + z.$$

  If $z < 0$, it would imply a blueshift. All distant galaxies show redshifts.
- When Hubble plotted the redshift z against the distance r, he found them to be proportional to each other $z = Kr$.
- Now, the redshift can be interpreted as arising from a Doppler velocity v which is given by

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}} \approx 1 + \frac{v}{c}$$

  for non-relativistic speeds $v \ll c$.
- Thus Hubble interpreted the law as

$$v = H_0 r,$$

  where $H_0 \equiv c\, K$ is a constant, known as **Hubble's constant**.
- Thus he concluded that distant galaxies move away from us with a speed which is proportional to its distance from us. This was known as the **Hubble's law**.

Lemaitre's work

- Georges Lemaitre published a paper in 1927 (two years before Hubble's results) where he measured the constant H_0 using the published data on distance and redshifts.
- The original paper was in French, later translated in English (1931).
- During its XXX (30th) General Assembly in Vienna (in August 2018), the International Astronomical Union (IAU) put forward a draft resolution to rename the Hubble law as the **Hubble-Lemaitre law**. The resolution was proposed to recognise Lemaitre's research on the expansion of the Universe, and to pay tribute to both Lemaitre and Hubble for their fundamental contributions to the development of modern cosmology.
- All Individual and Junior Members of the IAU (11072 individuals) were invited to participate in an electronic vote, following which *the resolution was accepted.*

Hubble-Lemaitre law versus Copernican principle

- It seems that the Hubble's law violates the concept of homogeneity; since all galaxies are moving away from us, it would seem that we are in some kind of a "centre of the Universe".
- This can be shown to be *untrue* if we write the law in the vector form

$$\vec{v} = H_0 \vec{r}.$$

  This equation is invariant under translation and rotation, thus ensuring that it is consistent with homogeneity and isotropy.
- To elaborate this point, let us assume the law holds true in the reference frame centred on us. Let us consider a galaxy at the coordinate $\vec{r}_1$ moving with a velocity $\vec{v}_1 = H_0 \vec{r}_1$ with respect to us.
- The observers at rest in the refernce frame of this galaxy will find the velocity field of other galaxies to be

$$\vec{v}' = \vec{v} - \vec{v}_1 = H_0 \left(\vec{r} - \vec{r}_1 \right) = H_0 \vec{r}',$$

  thus ensuring that the law holds in any other galaxy as well.
- This shows that Hubble's law does not violate homogeneity or the Copernican principle.

The Hubble parameter

- It turns out that the *only* possible velocity-position relation which is consistent with homogeneity and isotropy of the Universe is

$$\vec{v} = H(t)\vec{r}.$$

- We will later see that the quantity H is indeed a function of time in our Universe. Hence the appropriate term to describe it would be **Hubble parameter**. The value of $H(t)$ at present epoch is denoted by H_0.
- This generalized equation can be integrated to give

$$\vec{r}(t) = a(t)\vec{x}, \quad a(t) = \mathrm{e}^{\int \mathrm{d}t \, H(t)}.$$

  where $\vec{x}$ is a constant.
- Obviously

$$H(t) = \frac{\mathrm{d}\ln a}{\mathrm{d}t} = \frac{\dot{a}}{a}.$$

The scale factor

- A useful consequence of the Hubble-Lemaitre law $\vec{r}(t) = a(t)\vec{x}$ is that the distance between two fundamental observers (galaxies) $\vec{r}(t)$ can be factored into a time variable part $a(t)$ and a fixed part $\vec{x}$ (ignoring peculiar motion due to gravity).
- The part $\vec{x}$ depends on the pair of objects but not on the time, while $a(t)$ is the **cosmic scale factor** and applies to the whole Universe.
- The quantity $\vec{x}$ is called the **comoving distance** between the two galaxies.
- Coventionally, $a(t)$ is normalized such that at the present epoch $a(t_0) = 1$.
- Note that when $a(t) \to 0$, the distance between any two galaxies $\vec{r} \to 0$, irrespective of the value of the corresponding $\vec{x}$. This means that a sufficiently early times, all galaxies where condensed into a single "point", which forms the basis for the **Big Bang model** of cosmology.

The spacetime metric

- Consider a set of observers for whom the Universe seems isotropic (observers moving with respect to these observers will *not* find the distribution isotropic) – we call them "fundamental observers".
- Let us try to write the metric for such observers (with $c = 1$)

$$ds^2 = g_{ij} dx^i dx^j = g_{00} dt^2 + g_{0\alpha} dt dx^\alpha - \sigma_{\alpha\beta} dx^\alpha dx^\beta .$$

- Isotropy implies $g_{0\alpha} = 0$, or else on would have a preferred vector $v_\alpha = g_{0\alpha}$.
- The time coordinate appropriate for these fundamental observers would be the "cosmological time". In the rest frame of such observers, we put $dx^\alpha = 0$ and the proper time would be $ds = dt$, implying $g_{00} = 1$.
- We then have

$$ds^2 = dt^2 - \sigma_{\alpha\beta} dx^\alpha dx^\beta .$$

- Note that according the assumptions made till now, the fundamental observers would follow the trajectory $x^i(s)$ given by

$$x^0 = s, \quad x^1 = x^2 = x^3 = \text{constant},$$

with the four-velocity being given by

$$u^i \equiv \frac{dx^i}{ds} = (1, 0, 0, 0) .$$

Homogeneity and isotropy

- Now, let us apply the requirement of homogeneity and isotropy to our metric.
- Imagine a triangle drawn in the space at some time $t = t_1$. At some other time $t = t_2$, the triangle would have a different size. However isotropy means that the shape of the triangle should remain the same (i.e., the two triangles should be similar).
- Consider three points O, A, B which are infinitesimally separated (i.e., an infinitesimal triangle). Now, if these three points are fundamental observers, their coordinates x^μ would not change with time.
- We take O to be the origin of the coordinates. The distances to the other poinnts are

$$\mathrm{d}L_A^2(t) = \sigma_{\alpha\beta}\left(t, x_A^\mu\right) \mathrm{d}x_A^\alpha\ \mathrm{d}x_A^\beta, \quad \mathrm{d}L_B^2(t) = \sigma_{\alpha\beta}\left(t, x_B^\mu\right) \mathrm{d}x_B^\alpha\ \mathrm{d}x_B^\beta.$$

- Now because of isotropy, we expect the triangle to remain similar, i.e.,

$$\frac{\mathrm{d}L_A^2(t_2)}{\mathrm{d}L_A^2(t_1)} = \frac{\mathrm{d}L_B^2(t_2)}{\mathrm{d}L_B^2(t_1)} \Longrightarrow \frac{\sigma_{\alpha\beta}\left(t_2, x_A^\mu\right) \mathrm{d}x_A^\alpha\ \mathrm{d}x_A^\beta}{\sigma_{\alpha\beta}\left(t_1, x_A^\mu\right) \mathrm{d}x_A^\alpha\ \mathrm{d}x_A^\beta} = \frac{\sigma_{\alpha\beta}\left(t_2, x_B^\mu\right) \mathrm{d}x_B^\alpha\ \mathrm{d}x_B^\beta}{\sigma_{\alpha\beta}\left(t_1, x_B^\mu\right) \mathrm{d}x_B^\alpha\ \mathrm{d}x_B^\beta}.$$

- Now, homogeneity implies that this relation has to be true for any two points, hence we must have the ratio to be independent of x^μ. This, in turn, implies that we have to write

$$\sigma_{\alpha\beta}\left(t, x_A^\mu\right) \mathrm{d}x_A^\alpha\ \mathrm{d}x_A^\beta = S^2(t) h_{\alpha\beta}\left(x_A^\mu\right) \mathrm{d}x_A^\alpha\ \mathrm{d}x_A^\beta.$$

- The form of the spatial metric becomes

$$\sigma_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta = S^2(t) h_{\alpha\beta}(x^\mu)\mathrm{d}x^\alpha\mathrm{d}x^\beta.$$

The isotropic spatial metric

- To determine the 3-space line element at some given fixed time $\mathrm{d}l^2 = h_{\alpha\beta}(x^\mu)\mathrm{d}x^\alpha \mathrm{d}x^\beta$, note that isotropy implies the metric to depend only on the three rotational invariants $\vec{x}\cdot\vec{x}$, $\vec{x}\cdot\mathrm{d}\vec{x}$ and $\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}$.
- The natural coordinate to work will be the spherical polar coordinates r, θ, ϕ, where $r^2 \equiv \vec{x}\cdot\vec{x}$. The metric must take the form

$$\mathrm{d}l^2 = A(r)\,(\vec{x}\cdot\mathrm{d}\vec{x})^2 + B(r)\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} = A(r)r^2\mathrm{d}r^2 + B(r)\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right),$$

  or equivalently

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{C(r)} + D^2(r)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2),$$

  where $1/C(r) = A(r)r^2 + B(r)$ and $D^2(r) = r^2 B(r)$.
- We can always define a new radial coordinate $r' = D(r)$ so that the metric becomes (dropping the primes)

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2),$$

  where $f(r)$ is an arbitrary function of r. Note that we must have $f(r) = 1$ when there is no spatial curvature.

Homogeneity of space

- Homogeneity of space implies that the *spatial* scalar curvature ${}^3R$ should be independent of $\{r, \theta, \phi\}$.
- One can show that, for the above metric $\mathrm{d}l^2$, the scalar curvature is given by

$$ {}^3R = -\frac{2}{r^2}(-1 + f + rf'). $$

- Equating the above quantity to a constant $6K$ (which has dimensions of 1/length^2), we get

$$ rf' + f = 1 - 3Kr^2. $$

- We then have the solution of the differential equation as

$$ f = 1 - Kr^2 + \frac{C}{r}. $$

- As $K \to 0$, we require the space to be flat $f \to 1$, which sets the integration constant $C = 0$. Then the function is found to be

$$ f = 1 - Kr^2. $$

The FLRW metric

- Thus the spatial part of the metric becomes

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2),$$

  and the full metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - S^2(t)\left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right].$$

- This is known as the **Friedmann–Lemaitre–Robertson–Walker metric**.
- Note that the curvature K is a constant which has dimensions of $1/\mathrm{length}^2$. It can be either negative, positive or zero.
- Clearly r has dimensions of length and $S(t)$ is dimensionless.

Different forms of the metric: I

- It is possible to bring the above metric to a more familiar form by defining a new coordinate

$$r' = \begin{cases} r\sqrt{|K|} & \text{for } K \neq 0, \\ r & \text{for } K = 0, \end{cases}$$

  a new function

$$R^2(t) = \begin{cases} \dfrac{S^2(t)}{|K|} & \text{for } K \neq 0, \\ S^2(t) & \text{for } K = 0, \end{cases}$$

  and a dimensionless constant

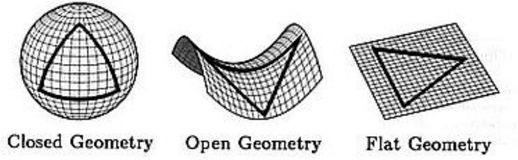
$$k = \frac{K}{|K|} = \begin{cases} +1 & \text{when } K > 0, \\ -1 & \text{when } K < 0, \\ 0 & \text{when } K = 0. \end{cases}$$

- The metric then becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right],$$

  where we have dropped the prime.
- Here r is dimensionless and $R(t)$ has dimensions of length.

Spatial curvature



- **Flat:** For $k = 0$, we have $\mathrm{d}l^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$, which is just the flat, Euclidean three-space.
- **Closed:** For $k = 1$, we have
$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1 - r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2),$$
which represents a three-sphere embedded in an abstract four-dimensional Euclidean space. The volume of the entire space is finite.
- **Open:** For $k = -1$, we have
$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1 + r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2),$$
which represents a three-dimensional hyperboloid embedded in an abstract four-dimensional Lorentzian space (should not be confused with the physical spacetime). The volume of the entire space is infinite in this case.
- Most recent observations seem to indicate that $k = 0$.

Different forms of the metric: II

- Another form of the metric can be written by introducing a different dimensionless coordinate

$$\chi = \int \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} = \frac{\sin^{-1}\left(r\sqrt{k}\right)}{\sqrt{k}} \equiv S_k^{-1}(r),$$

where

$$S_k^{-1}(r) \equiv \frac{\sin^{-1}\left(r\sqrt{k}\right)}{\sqrt{k}} = \begin{cases} \sin^{-1} r & \text{if } k = 1 \\ r & \text{if } k = 0 \\ \sinh^{-1} r & \text{if } k = -1. \end{cases}$$

- Then the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left[\mathrm{d}\chi^2 + S_k^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right].$$

- The area of the surface of a sphere in terms of the new coordinates is $4\pi R^2(t) S_k^2(\chi) = 4\pi R^2(t) r^2$.