Cosmology: Exercise Sheet 1

IUCAA-NCRA Graduate School

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1. It is possible to show that the Hubble's law $\boldsymbol{v} = H_0 \boldsymbol{r}$ is the only law that is consistent with homogeneity and isotropy of the universe.

   (i) Assume that the velocity field of galaxies in the reference frame of our galaxy is given by

   $$\boldsymbol{v} = \boldsymbol{f}(\boldsymbol{r}),$$

   where $\boldsymbol{f}$ is a unknown vector function to be determined. Show that the only function which is consistent with the principle of homogeneity is a linear one, i.e., $f_a(\boldsymbol{r}) = \sum_{b=1}^{3} H_{ab}\ r_b,\ a = 1, 2, 3.$

   (ii) Next show that isotropy implies $H_{ab} = H\delta_{ab}$.

2. According to Weyl's postulate, the timelike worldlines of fundamental observers form a three-bundle of non-intersecting geodesics orthogonal to a series of spacelike hypersurfaces.

   (i) If we use three spacelike coordinates $x^\alpha =$ constant to label the worldline of a fundamental observer, and $x^0 = t =$ constant to describe a spacelike hypersurface, then show that the orthogonality condition implies

   $$g_{ik} u^i a^k = 0,$$

   where $u^i = \mathrm{d}x^i/\mathrm{d}s$ is the four-velocity of the fundamental observer and a^k is a arbitrary vector on the hypersurface.

   (ii) Argue that the two vectors can be taken as $u^i = (u^0, 0, 0, 0)$ and $a^i = (0, \boldsymbol{a})$. Then show that the orthogonality condition becomes $g_{0\alpha} = 0$.

   (iii) Use the geodesic equation to show that g_{00} must be a function of only $x^0 = t$.

   (iv) Next define a new time coordinate $t \to \int \mathrm{d}t\ \sqrt{g_{00}(t)}$ and write the form of the resultant metric.

3. (i) Consider a four-dimensional Euclidean space with a metric

   $$\mathrm{d}L^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2.$$

   A three-sphere of radius R in this space can be described by the equation

   $$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2.$$

   What is the metric on the surface of the sphere? What is the volume of the three-sphere?

   (ii) Consider a four-dimensional Lorentzian space with a metric

   $$\mathrm{d}L^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \mathrm{d}x_4^2.$$

   A three-hyperboloid in this space can be described by the equation

   $$x_4^2 - x_1^2 - x_2^2 - x_3^2 = R^2.$$

   What is the metric on the surface of the hyperboloid? What is the volume of the hyperboloid?

4. Calculate the Ricci scalar for the metric

   $$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{f(r)} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2\right).$$

5. Find the series expansions for the proper distance $d_P(z)$, the comoving distance $d_C(z)$, the angular-diameter distance $d_A(z)$ and the luminosity distance $d_L(z)$. Keep the first two non-zero terms in each case.

6. Consider a matter component having a time-varying equation of state $P(a) = w(a)\rho(a)$. Use the energy conservation equation to show that

$$\rho(a) = \frac{\rho_0}{a^3} \exp\left[3\int_a^1 \frac{\mathrm{d}a'}{a'} w(a')\right].$$

7. Consider an monatomic ideal gas consisting of particles of mass m.

(i) Show that the pressure is given by

$$P = \frac{k_B T}{m}\rho_m.$$

Note that $w \neq k_B T/m$ because $\rho \neq \rho_m$. There is a kinetic energy term which needs to be accounted for.

(ii) For a gas with adiabatic index γ, we can write

$$P = (\gamma - 1)(\rho - \rho_m)$$

where we have accounted for the fact that the pressure for a non-relativistic gas is contributed only by the kinetic energy. Then show that the equation of state is given by

$$w = \frac{k_B T}{m}\left[1 + \frac{1}{\gamma - 1}\frac{k_B T}{m}\right]^{-1}.$$

Show that for a non-relativistic gas $w \ll 1$.

8. Consider a universe consisting only of non-relativistic particles with negligible pressure. Let the density parameter of this matter at the present epoch be Ω_0. Find the scale factor $a(t)$ for the cases $k = +1, 0, -1$. Also work out the expressions for the age of the universe.