Astrophysics: Final Examination

HRI Graduate School
August - December 2011

05 December 2011
Duration: 3 hours

- The paper is of 60 marks. Attempt all the questions.
- You are free to consult your class notes during the examination.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. (i) Solve the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n; \qquad \theta(0) = 1, \qquad \left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi=0} = 0$$

for $n = 0$. Find the form of $\theta(\xi)$ and the value of ξ_1 (defined so that $\theta(\xi_1) = 0$).

(ii) Show that

$$\theta = \frac{1}{\sqrt{1 + \xi^2/3}}$$

is a solution of the Lane-Emden equation. What would be the corresponding value of n? Does this solution satisfy the boundary conditions mentioned in part (i)?

[4 + 3]

2. Consider a source of radiation that moves with a speed v at an angle θ to the line of sight.

(i) Show that the apparent velocity $v_\perp$ of the source will be given by

$$v_\perp = \frac{v\ \sin\theta}{1 - (v/c)\cos\theta}$$

(ii) Show that for a given value of v, the apparent velocity $v_\perp$ is maximum when $\theta = \cos^{-1}(v/c)$, with the maximum value being γv where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$.

(iii) Show that in the limit $v \to c$, the maximum occurs for $\theta \approx \gamma^{-1}$.

(iv) Show that the necessary condition for superluminal motion to occur (at some angle) is $v > c/\sqrt{2}$.

[5 + 3 + 2 + 2]

3. Show that the evolution of the radius $R(t)$ of a HII region around a star is of the form

$$R(t) = r_s\left(1 - \mathrm{e}^{-t/t_{\rm rec}}\right)^{1/3}$$

where r_s is the Stromgren radius and $t_{\rm rec}$ is the recombination time-scale. What is the form of $t_{\rm rec}$ and what would its value for ISM? Assume the luminosity of the star to be constant and the temperature of the HII region to be uniform.

[3]

4. (i) Show that, for a Friedmann-Robertson-Walker universe, the age is related to the redshift by the relation

$$t = \int_z^\infty \frac{\mathrm{d}z'}{(1+z')H(z')}$$

(ii) Show that, for a flat universe filled with only one kind of energy component with an equation of state $P = w\rho$, the age-redshift relation reduces to

$$t = \frac{2}{3(1+w)H_0}(1+z)^{-3(1+w)/2}$$

Hence show that the age is related to the density through

$$t = \sqrt{\frac{1}{6(1+w)^2\pi G\rho}}$$

[3 + 3]

5. (i) Show that the collisionless Boltzmann equation for an expanding homogeneous and isotropic universe is given by

$$\frac{\partial f}{\partial t} - H(t)\, p\frac{\partial f}{\partial p} = 0$$

(ii) Integrate the above equation over all momenta to obtain the evolution equation for number density n of particles. Hence show that $n \propto a^{-3}$.

(iii) Similarly, integrate the equation over $\int \mathrm{d}^3p\ E(p)$ to obtain the energy conservation equation.

[3 + 4 + 4]

6. Consider a particle moving along a geodesic with $\theta =$ constant and $\phi =$ constant in a FRW universe.

(i) Show that the zeroth component of the geodesic equation reads as

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \frac{a\dot{a}}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = 0$$

(ii) Now, use the condition $g_{ik}U^iU^k = 1$ to show that

$$\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \frac{a^2}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = 1$$

(iii) Eliminate $\mathrm{d}r/\mathrm{d}s$ from the two equations to obtain a differential equation for $t(s)$. Then integrate the differential equation and show that the solution is

$$a^2\left[\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - 1\right] = \text{constant}$$

(iv) Use the above to show that the magnitude of the three-momentum of the particle varies as a^{-1}.

[4 + 2 + 4 + 3]

7. (i) Consider a collection of particles attracting each other through gravity. If this collection is in a steady state, show that

$$2\overline{T} + \overline{V} = 0$$

where

$$T = \sum_i \frac{1}{2}m_i\dot{\mathbf{x}}_i^2; \quad V = -\sum_{i,j>i}\frac{Gm_im_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

and the average is obtained by integrating over a sufficiently long time τ and dividing all the terms by τ.

(ii) Using the equation for hydrostatic equilibrium, show that the virial theorem

$$2E_T + E_G = 0$$

holds within stars. Here E_G is the total gravitational energy of the star and E_T is the total thermal (kinetic) energy.

[5 + 3]