Astrophysics: Assignment 8

HRI Graduate School

August - December 2011

10 November 2011

To be returned to the tutor by 26 November 2011

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Metric in a uniformly rotating frame:** The special relativistic metric is $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$. Consider a frame rotating uniformly around the z axis with respect to this frame such that

$$t' = t, \quad z' = z, \quad x' = x\cos\Omega t + y\sin\Omega t, \quad y' = -x\sin\Omega t + y\cos\Omega t$$

Find out the metrics g_{ik} in the rotating frame.

[4]

2. **Specific intensity in expanding universe:** Let $f(t, \mathbf{x}, \mathbf{p})$ be the phase-space distribution of photons which is conserved during propagation without interaction.

(i) Show that $f(t, \mathbf{x}, \mathbf{p})$ can be written as

$$f(t, \mathbf{x}, \mathbf{p}) = \frac{\mathrm{d}N_\gamma}{\mathrm{d}^3x\ \mathrm{d}^3p} \propto \frac{\mathrm{d}N_\gamma}{c\ \mathrm{d}t\ \mathrm{d}A\ \nu^2\ \mathrm{d}\nu\ \mathrm{d}\Omega}$$

where $\mathrm{d}\Omega$ is a solid angle around the direction of propagation, $\mathrm{d}A$ is the area normal to the direction of propagation and other symbols have their usual meanings.

(ii) Recall that the specific intensity is defined as

$$I_\nu = \frac{\mathrm{d}E_\gamma}{\mathrm{d}t\ \mathrm{d}A\ \mathrm{d}\nu\ \mathrm{d}\Omega}$$

Show that the quantity I_ν/ν^3 is invariant during photon propagation. Hence show that $I_\nu \propto a^{-3}$ in an expanding universe.

(iii) The total energy density of (isotropic) radiation is defined as

$$u = \int \mathrm{d}\nu\ u_\nu = \frac{4\pi}{c}\mathrm{d}\nu\ I_\nu$$

Show that $u \propto a^{-4}$ in an expanding universe.

[1 + 2 + 2]

3. **Friedmann equations:** Show that the Friedmann equations obtained using ${}^0_0$ and ${}^\alpha_\alpha$ components of Einstein equations can be combined to give the energy conservation equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3\frac{\dot{R}}{R}(\rho + P) = 0$$

[3]

4. **deSitter line element** (i) Solve Einstein equations for $k = 1$ and $k = 0$ when the equation of state for matter is $P = -\rho$. Show that the line elements have the form

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}t^2 - \mathrm{e}^{2Ht}\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right] \\ \mathrm{d}s^2 &= \mathrm{d}T^2 - \frac{1}{H^2}\cosh^2 HT\left[\frac{\mathrm{d}R^2}{1-R^2} + R^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right] \end{aligned}$$

respectively.

(ii) Show that the above two line elements represent the same spacetime by finding the coordinate transformation between (t, r) and (T, R).

(iii) Is the spacetime is spatially flat or closed? Comment on this.

[3 + 6 + 2]

5. **Mattig's formula:** Show that, for a matter-dominated universe (not necessarily flat), the comoving distance to an object at redshift z is given by

$$R_0\chi = \frac{1}{H_0}\int_0^z \frac{\mathrm{d}z'}{(1+z')\sqrt{1+\Omega_0 z'}}$$

Hence show that the quantity (which appears in the expressions for $d_L(z)$ and $d_A(z)$ is

$$R_0 S_k(\chi) = \frac{2}{H_0}\frac{\Omega_0 z + (\Omega_0 - 2)\left[\sqrt{1+\Omega_0 z} - 1\right]}{\Omega_0^2(1+z)}$$

[9]

6. **Varying equation of state:** Consider a matter component having a time-varying equation of state, i.e., $P(a) = w(a)\rho(a)$. Show that the density of such a form of matter evolves as

$$\rho(a) = \frac{\rho_0}{a^3}\exp\left[3\int_a^1 \frac{\mathrm{d}a'}{a'}w(a')\right]$$

[3]

7. **General relativistic form of the Boltzmann equation:** Show that the general relativistic form of the Boltzmann equation for particles affected by gravitational forces and collisions is

$$U^i\frac{\partial f}{\partial x^i} - \Gamma^i_{jk}U^jU^k\frac{\partial f}{\partial U^i} = C[f]$$

[3]

8. **Chemical potential of electrons and positrons:** (i) At temperatures $T \ll m_e$, the equilibrium between electrons (e^-), positrons (e^+) and photons (γ) is maintained by the reaction $e^- + e^+ \leftrightarrow \gamma + \gamma$. Hence show that the chemical potentials of electron and positron are related by $\mu_{e^-} = -\mu_{e^+}$.

(ii) Show that, for $T \gg m_e$, the excess of electrons over positrons is then

$$n_{e^-} - n_{e^+} \approx \frac{g_e T^3}{6\pi^2}\left[\pi^2\left(\frac{\mu_e}{T}\right) + \left(\frac{\mu_e}{T}\right)^3\right]$$

where $\mu_e \equiv \mu_{e^-}$. As the universe cools to temperature $T \ll m_e$, electrons and positrons will annihilate in pairs and only this small excess will survive.

(iii) The only other charged particle that will be present in the universe is the proton. Because our universe appears to be electrically neutral, we expect $n_{e^-} - n_{e^+} = n_p$. Assuming that the baryon density in the universe is contributed only by protons, show that $n_p/n_\gamma \sim 10^{-8}$.

(iv) Hence show that $\mu_e/T \sim 10^{-8}$.

[1 + 6 + 3 + 2]