Astrophysics: Assignment 6

HRI Graduate School
August - December 2011

20 October 2011
To be returned to the tutor by 03 November 2011

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Motion in a spherically symmetric potential:** Consider the motion of a particle in a spherically symmetric potential $\Phi(r)$.

   (i) Show that the energy (per unit mass) E and the angular momentum (per unit mass) $\mathbf{L}$ of the particle are conserved. Hence argue that the motion is confined to a plane.

   (ii) Write down the expressions for E and $L = |\mathbf{L}|$ in terms of the polar coordinates r, ψ in the plane of motion.

   (iii) Assume the motion to be bound. Then the motion will be contained within the turning points r_1 and $r_2 \geq r_1$. What is the condition for obtaining these turning points?

   (iv) The radial period T_r is the time required for the star to travel from r_1 to r_2 and back. Show that

   $$T_r = 2 \int_{r_1}^{r_2} \frac{\mathrm{d}r}{\sqrt{2[E - \Phi(r)] - \frac{L^2}{r^2}}}$$

   (v) Show that, in travelling from r_1 to r_2 and back, the azimuthal angle ψ changes by an amount

   $$\Delta\psi = 2L \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r^2\sqrt{2[E - \Phi(r)] - \frac{L^2}{r^2}}}$$

   (vi) Hence show that the azimuthal period T_ψ is related to the radial period by

   $$T_\psi = \frac{2\pi}{\Delta\psi} T_r$$

   $[3 + 2 + 1 + 2 + 2 + 1]$

2. **Time periods for the isochrone potential:** Show that, for the isochrone potential

   $$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}},$$

   the radial time period is given by

   $$T_r = \frac{2\pi GM}{(-2E)^{3/2}}$$

   while the change in azimuthal angle during this period is given by

   $$\Delta\psi = \pi \left(1 + \frac{L}{\sqrt{L^2 + 4GMb}}\right)$$

   What happens to $\Delta\psi$ when $b \to 0$ and $b \to \infty$?

   [8]

3. **Virial theorem for stellar clusters:** Consider a collection of particles attracting each other through gravity. If this collection is in a steady state, show that

   $$2\overline{T} + \overline{V} = 0$$

   where

   $$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{x}}_i^2; \quad V = -\sum_{i,j>i} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

   and the average is obtained by integrating over a sufficiently long time τ and dividing all the terms by τ.

   [5]