Astrophysics: Assignment 4

HRI Graduate School
August - December 2011

10 September 2011
To be returned to the tutor by 22 September 2011

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Importance of general relativistic effects in white dwarfs:** For typical masses and sizes of white dwarfs, estimate whether general relativistic effects could be important. For stars having masses $\sim 1M_\odot$, estimate the radius and density when general relativistic effects could be important, say, $GM/c^2R \sim 0.1$.

   [2]

2. **Field of a magnetic dipole:** Assume a system of charges with current which is steady state, i.e., the bulk properties of the system do not change with time.

   (i) Show that for such system, the current obeys the equation $\nabla \cdot \mathbf{J} = 0$. Write down the Maxwell equations for the magnetic field $\mathbf{B}$.

   (ii) Show that under Coulomb gauge, the equations reduce to

   $$\nabla^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J}$$

   (iii) The solution of the above is analogous to electrostatic potential. Write down the solution in an integral form. If the current is localized in a region and we are observing at a distance far from the source, we can carry out a multipole expansion. Write the first two terms of the expansion.

   (iv) Show that for a localized current, the first (monopole) term vanishes.

   (v) For the dipole term, show that the magnetic potential is

   $$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3}$$

   where the magnetic moment is defined as

   $$\mathbf{m} = \frac{1}{2}\int \mathrm{d}^3x' \ \mathbf{x}' \times \mathbf{J}(\mathbf{x}')$$

   Hence show that the magnetic field of a dipole is given by

   $$\mathbf{B}(\mathbf{x}) = \frac{1}{c}\left(\frac{3(\mathbf{m}\cdot\mathbf{x})\mathbf{x}}{|\mathbf{x}|^5} - \frac{\mathbf{m}}{|\mathbf{x}|^3}\right)$$

   (vi) What is the magnetic field in the direction of the dipole moment?

   [2 + 2 + 3 + 2 + 4 + 1]

3. **Mass determination using binary stars:**

   (i) Consider two stars in orbit about their centre of mass. Assuming that the orbital plane is perpendicular to the observer's line of sight (i.e., the orbital plane is parallel to the plane of the sky), show that the mass ratio is given by

   $$\frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1}$$

   where α_1, α_2 are the angles subtended by the semi-major axes of the elliptic orbits of the two stars respectively. We can use this to measure the mass ratios for visual binaries.

(ii) Show that the orbital period is given by

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3$$

where $a = a_1 + a_2$ is the sum of the semi-major axes. You can recognize the above as Kepler's third law, derive it starting from the Lagrangian of the system. Suppose we have measured α_1, α_2 and P, then what else do we need to measure so as to estimate the masses of each star?

(iii) Now, let i be the angle of inclination between the plane of the orbit and the plane of the sky. Show that the observer will not measure the actual angles subtended by the semi-major axes α_1 and α_2 but their projections onto the plane of the sky, $\tilde{\alpha}_1 = \alpha_1 \cos i$ and $\tilde{\alpha}_2 = \alpha_2 \cos i$. Hence argue that this geometrical effect plays no role in calculating the mass ratios m_1/m_2.

(iv) However, this projection effect can make a significant difference when we are using Kepler's third law. Show that Kepler's third law may be solved for the sum of the masses to give

$$m_1 + m_2 = \frac{4\pi^2}{G} \left(\frac{d}{\cos i} \right)^3 \frac{\tilde{\alpha}^3}{P^2}$$

where d is the distance to the system and $\tilde{\alpha} = \tilde{\alpha}_1 + \tilde{\alpha}_2$. Thus one can measure the individual masses from radial velocities if i and d are known.

(v) Often it is not possible to resolve the binary systems and hence it is not possible to measure α_1, α_2. In such cases, one can still measure the velocities of the individual stars using spectroscopy.

If we assume that the orbital eccentricity is very small, then orbit can be taken to circular. In this case, the speeds of the stars are essentially constant. Show that the ratio of the masses of the two stars becomes

$$\frac{m_1}{m_2} = \frac{v_2}{v_1}$$

(vi) Unfortunately, one often is able to measure only the component of the velocity along the line of sight (using Doppler methods). Show that this component is related to the actual velocity by $v_{1r} = v_1 \sin i$ and $v_{2r} = v_2 \sin i$. Then what is m_1/m_2? As is the situation with visual binaries, we can determine the ratio of the stellar masses without knowing the angle of inclination.

(vii) Show that the sum of masses is related to the orbital period is given by

$$m_1 + m_2 = \frac{P}{2\pi G} \frac{(v_{1r} + v_{2r})^3}{\sin^3 i}$$

(viii) In some cases, one star is much brighter than its companion and hence the spectrum of the dimmer member will be overwhelmed (astrometric binary). If the spectrum of star 1 is observable but the spectrum of star 2 is not, then one can estimate only v_{1r}. For such systems, show that

$$\frac{m_2^3}{(m_1 + m_2)^2} \sin^3 i = \frac{P}{2\pi G} v_{1r}^3$$

This is known as the **mass function** of the system. The mass function is useful only for statistical studies or if an estimate of the mass of at least one component of the system already exists by some indirect means. If either m_1 or $\sin i$ is unknown, the mass function sets a lower limit for m_2, since the left-hand side is always less than m_2.

$[3 + 4 + 2 + 2 + 1 + 2 + 3 + 2]$