Astrophysics: Assignment 3

HRI Graduate School

August - December 2011

03 September 2011

To be returned to the tutor by 15 September 2011

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Threshold temperature for nuclear fusion:** Nuclear fusion of two hydrogen nuclei (protons) occur when they come close and "stick together" to form a deuterium nucleus. The protons have to overcome the electrostatic Coulomb repulsive force and come within a distance of $\sim r_0 = 10^{-13}$ cm at which point the attractive "strong" force takes over.

   (i) Assume the height of the Coulomb barrier to be e^2/r_0. Classically, the protons can overcome this if they have kinetic energies larger than the barrier height. For thermal motions, the average kinetic energy is $\sim k_B T$, hence the condition for nuclear fusion to occur is $k_B T > e^2/r_0$. Estimate this threshold temperature.

   (ii) It is found that fusion can commence when $T \gtrsim 10^7$ K. Does your result match with this finding? What has possibly gone wrong?

   [2 + 1]

2. **The virial theorem for stars:** Using the equation for hydrostatic equilibrium, show that the virial theorem

   $$E_T = -\frac{1}{2} E_G = \frac{1}{2} |E_G|$$

   holds within stars. Here E_G is the total gravitational energy of the star and E_T is the total thermal (kinetic) energy.

   [3]

3. **Analytical solutions of the Lane-Emden equation:** Solve the Lane-Emden equation

   $$\frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right) = -\theta^n; \qquad \theta(0) = 1, \qquad \left. \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right|_{\xi=0} = 0$$

   for $n = 0, 1$. Find the form of $\theta(\xi)$ and the value of ξ_1 (defined so that $\theta(\xi_1) = 0$) in each case.

   [6]

4. **Mean molecular weight:** (i) What is the mean molecular weight of ionized hydrogen?

   (ii) Suppose a gas contains a mixture of neutral hydrogen and helium. The mass fraction of helium is Y. What is the mean molecular weight of this composition?

   (iii) Suppose a gas is completely ionized (as is expected within stellar interiors with very high temperatures) and consists of hydrogen (with mass fraction X), helium (with mass fraction Y) and other heavier metals (with mass fraction Z). Obviously $X + Y + Z = 1$. Show that the mean molecular weight in this case is

   $$\mu = \left(2X + \frac{3}{4} Y + \frac{1}{2} Z \right)^{-1}.$$

   You can assume that a heavier element with atomic number A has A neutrons and that $A \gg 1$.

   [1 + 2 + 3]

5. **Importance of radiation pressure in stars:** Show that the radiation pressure at the centre of the Sun is negligible compared to the gas pressure, by estimating the ratio of the radiation pressure to the gas pressure.

   [2]

6. **Eddington model for high mass stars:** (i) Consider a star in which gas pressure and radiation pressure are both important (i.e., the total pressure is the sum of the two). If the gas pressure is equal to a constant fraction β of the total pressure everywhere inside the star, then show that the total pressure has to be related to the density in the following way

$$P = \left(\frac{3k_B^4}{a_B \mu^4 m_p^4}\right)^{1/3} \left(\frac{1-\beta}{\beta^4}\right)^{1/3} \rho^{4/3}$$

(ii) Now consider several stars with different masses having the same composition (i.e. the same μ). Assuming that inside each of these stars the gas pressure is everywhere a constant fraction β of the total pressure (but β has different values for different stars), show that β inside a star would be related to its mass M by an equation of the form

$$\frac{1-\beta}{\beta^4} = CM^2$$

where C is a constant which you have to evaluate. Show that β is smaller for larger M, implying that radiation pressure is increasingly more important inside more massive stars. This is a historically important argument first given by Eddington.

(iii) Calculate the value of $M/M_\odot$ for which $\beta = 0.5$ (i.e., the radiation pressure is equal to the gas pressure). Also calculate the value of β for the sun. Assume $\mu \approx 0.62$ (appropriate for solar composition).

[3 + 2 + 2]

7. **Eddington luminosity:** The radiation pressure becomes more important inside more massive stars. Eventually the very high radiation pressure inside a massive star can make the star unstable. To be more specific, if the outward continuum radiation pressure exceeds the inward gravitational force, the star will not be able to hold on to its outer layer which would be blown away by radiation. Show that this condition leads to an upper limit on the luminosity of stars (known as the Eddington luminosity). Write the expression for the Eddington luminosity.

Using the fact that the opacity in very hot stars is provided by Thomson scattering, show that L/M has to be less than a critical value and find its value in terms of $L_\odot/M_\odot$. Assume that the only element present in the star is hydrogen.

[4]

8. **Jeans mass from virial theorem:** Consider a spherical cloud having a constant density ρ and a constant temperature T. Assume the cloud to have a mass M and radius R.

(i) Write down the expressions for the gravitational potential energy E_G and the thermal kinetic energy E_T of the cloud.

(ii) According to virial theorem, the cloud will be stable if $2E_T = -E_G$. Hence the condition for collapse would be $-E_G > 2E_T$. Use this condition to obtain the expression for Jeans mass:

$$M_J = \left(\frac{3}{4\pi\rho}\right)^{1/2} \left(\frac{5k_B T}{G\mu m_p}\right)^{3/2}$$

(iii) In the interstellar medium, we typically have $\rho \sim 10^{-24}$ gm cm^{-3} and $T \sim 100$ K. Estimate the value of the Jeans mass. Take $\mu = 1$.

[2 + 3 + 1]