PHY-T311 ASTRONOMY AND ASTROPHYSICS-I: Assignment 2

Department of Physics
Savitribai Phule Pune University
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To be returned in the class on 5 September 2018

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Radiative equilibrium:**

(i) Start with the radiative transfer equation

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}t} \equiv \frac{\partial I_\nu}{\partial t} + c\;\hat{n}\cdot\vec{\nabla}_x I_\nu = -c\kappa_\nu I_\nu + c j_\nu,$$

integrate it over all solid angles and show that the result can be written as

$$\frac{\mathrm{d}u_\nu}{\mathrm{d}t} \equiv \frac{\partial u_\nu}{\partial t} + \vec{\nabla}_x\cdot\vec{F}_\nu = 4\pi j_\nu - c\kappa_\nu\;u_\nu,$$

which has the form of a "conservation equation" (i.e., change in density + divergence of flux = sources − sinks). Remember that

$$u_\nu = \frac{1}{c}\int \mathrm{d}\Omega\;I_\nu,$$

and

$$\vec{F}_\nu = \int \mathrm{d}\Omega\;I_\nu\;\hat{n}.$$

(ii) The condition of *radiative equilibrium* is often expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \mathrm{d}\nu \int \mathrm{d}\Omega\;I_\nu\right) = 0.$$

Show that this is equivalent to

$$\int \mathrm{d}\nu\;\;(4\pi j_\nu - c\kappa_\nu\;u_\nu) = 0.$$

which implies that radiation should not add or subtract a net amount of energy from the system.

Write the above relation in terms of the source function S_ν.

What happens when κ_ν is independent of the frequency ν?

[3+4]

2. **Derivation of Saha equation:** Consider a system of quantum particles (could be fermions or bosons) called A in equilibrium at a temperature T.

(i) Write down the phase space distribution function $f_A(t, \vec{x}, \vec{p})$ if A is a fermion. Write down $f_A(t, \vec{x}, \vec{p})$ when A is a boson.

(ii) Show that the number density of such particles is given by

$$n_A(t, \vec{x}) \equiv \int \mathrm{d}^3p \; f_A(t, \vec{x}, \vec{p}) = \frac{4\pi}{c^3} \int_{m_A c^2}^{\infty} \mathrm{d}E \; E \; \sqrt{E^2 - m_A^2 c^4} \; E \; f_A.$$

(iii) Show that, in the non-relativistic limit ($k_B T \ll m_A c^2$), the expression reduces to (both for fermions and bosons)

$$n_A = \frac{4\pi g_A c^3}{h^3} \; m_A^3 \; \mathrm{e}^{\mu_A / k_B T} \int_1^{\infty} \mathrm{d}y \; \sqrt{y^2 - 1} \; \mathrm{e}^{-y m_A c^2 / k_B T}$$

You may assume $k_B T \ll m_A c^2 - \mu_A$ so that the system is "dilute" (i.e., the occupation numbers are much smaller than unity).

The integral in the non-relativistic limit is given by (you do *not* have to show this)

$$n_A = g_A \left(\frac{2\pi m_A k_B T}{h^2} \right)^{3/2} \mathrm{e}^{(\mu_A - m_A c^2)/k_B T}$$

(iv) Now consider the equilibrium system $p + e \rightleftharpoons \mathrm{H} + \gamma$ (photoionization and recombination). Consider three species $A = e, p, \mathrm{H}$. Write down the expressions for n_e, n_p and n_H.

(v) Finally, show that

$$\frac{n_p \; n_e}{n_H} = \frac{2 g_p}{g_H} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} \mathrm{e}^{-B/T}$$

where $B = (m_p + m_e - m_H) c^2$ is the binding energy of hydrogen atom.

[2+2+3+2+4]