PHY-T311 ASTRONOMY AND ASTROPHYSICS-I: Assignment 1

Department of Physics
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July – December 2018

7 August 2018
To be returned in the class on 17 August 2018

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Time periods for the isochrone potential:** Show that, for the isochrone potential

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}},$$

the radial time period is given by

$$T_r = \frac{2\pi GM}{(-2E)^{3/2}},$$

while the change in azimuthal angle during this period is given by

$$\Delta\psi = \pi \left(1 + \frac{L}{\sqrt{L^2 + 4GMb}}\right).$$

What happens to $\Delta\psi$ when $b \to 0$ and $b \to \infty$?

[10]

2. **Dynamical friction:** We consider the motion of a body of mass M through a population of stars of individual mass m. We first study the effect on M of an encounter with a single star, and then add the effects of successive encounters with different stars. A single encounter can be described by the motion of the reduced mass in the centre of mass frame.

(i) Let $(\vec{x}_M, \vec{v}_M)$ and $(\vec{x}_m, \vec{v}_m)$ be the positions and velocities of M and m, respectively; let $\vec{x} = \vec{x}_m - \vec{x}_M$ and $\vec{v} = \dot{\vec{x}}$. Then, show that the trajectory of the reduced mass μ is given by

$$\frac{1}{r} = C\cos(\psi - \psi_0) + \frac{G(m+M)}{L^2}, \qquad C = \sqrt{\frac{2E}{\mu L^2} + \frac{G^2(m+M)^2}{L^4}},$$

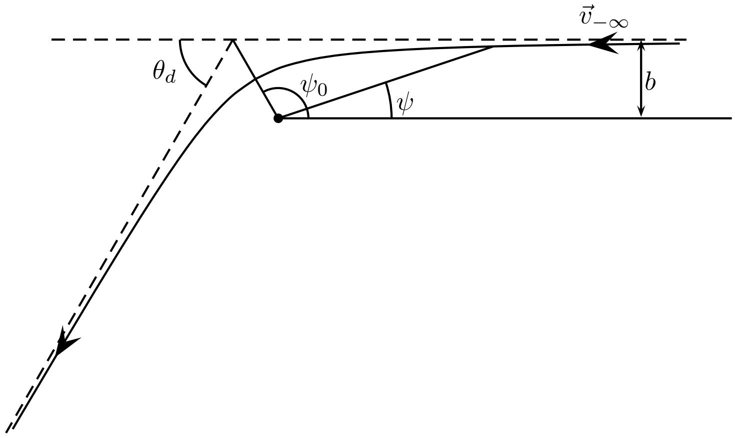
where $\vec{x} = (r, \psi)$ in polar coordinates in the plane of motion, E is the energy and L is the magnitude of the angular momentum.

(ii) Let the component of the initial separation vector that is perpendicular to the initial velocity vector $\vec{v}_{-\infty} = \vec{v}(t = -\infty)$ have length b (see the figure). We call b the **impact parameter** of the encounter. What are the values of L and E at the initial time $t \to -\infty$?

Hence show that

$$C = \frac{1}{b}\sqrt{1 + \frac{G^2(m+M)^2}{b^2 v_{-\infty}^4}}.$$

What is the magnitude of the velocity at $t \to \infty$?



(iii) For what value of ψ does the trajectory reaches the point of closest approach to the centre of mass?

(iv) Show that

$$\begin{aligned}\cos\psi_0 &= -\frac{G(m+M)}{\sqrt{b^2v_{-\infty}^4+G^2(m+M)^2}},\\ \sin\psi_0 &= \frac{bv_{-\infty}^2}{\sqrt{b^2v_{-\infty}^4+G^2(m+M)^2}},\\ \tan\psi_0 &= -\frac{bv_{-\infty}^2}{G(m+M)}.\end{aligned}$$

(v) If $\Delta\vec{v}_m$ and $\Delta\vec{v}_M$ are the changes in the velocities of m and M during the encounter, show that

$$\Delta\vec{v}_m = \frac{M}{m+M}\Delta\vec{v}, \quad \Delta\vec{v}_M = -\frac{m}{m+M}\Delta\vec{v}.$$

(vi) Show that the angle of deflection θ_d (see the figure) is related to ψ_0 by

$$\theta_d = 2\psi_0 - \pi.$$

(vii) Let $\Delta v_{M\parallel}$ and $\Delta v_{M\parallel}$ be the change in the velocity of M in the direction parallel and perpendicular to $\vec{v}_{-\infty}$, respectively. Show that

$$\begin{aligned}\Delta v_{M\parallel} = -\frac{m}{m+M}\Delta v_\parallel &= -2v_{-\infty}\frac{G^2m(m+M)}{b^2v_{-\infty}^4+G^2(m+M)^2},\\ \Delta v_{M\perp} = -\frac{m}{m+M}\Delta v_\perp &= 2v_{-\infty}^3\frac{Gmb}{b^2v_{-\infty}^4+G^2(m+M)^2}.\end{aligned}$$

(viii) Show that for $b \gg G(m+M)/v_{-\infty}^2$, if we keep terms upto $\mathcal{O}(b^{-1})$, we get

$$\Delta v_\parallel \to 0, \qquad \Delta v_\perp \approx \frac{2Gm}{bv_{-\infty}}.$$

(ix) If the star of mass M travels through an infinite homogeneous sea of stars, argue that the changes $\Delta\vec{v}_{M\perp}$ would sum to zero, while the changes $\Delta\vec{v}_{M\parallel}$ would form a non-zero resultant. In consequence, the mass M suffers a steady deceleration, which is said to be due to **dynamical friction**.

[4+2+2+3+2+1+3+2+1]