ASTRONOMY AND ASTROPHYSICS-I: Assignment 3

Department of Physics
Savitribai Phule Pune University
July – December 2019

15 October 2019
To be returned in the class on 24 October 2019

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

---

1. **The vorticity equation:**

   (i) Show that the Euler equation can be written as

   $$\frac{\partial \vec{V}}{\partial t} + + \frac{1}{2}\vec{\nabla} V^2 - \vec{V} \times \vec{\Omega} = -\frac{1}{\rho}\vec{\nabla} P + \vec{g},$$

   where $\Omega = \vec{\nabla} \times \vec{V}$ is the **vorticity**.

   (ii) Show, from the above equation, that the vorticity evolves as (assuming the external force $\vec{g}$ to be conservative)

   $$\frac{\partial \vec{\Omega}}{\partial t} = \vec{\nabla} \times (\vec{V} \times \vec{\Omega}) + \frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla} P.$$

   (iii) For an incompressible fluid, the density ρ does not change. Show that such a fluid obeys the equations

   $$\begin{aligned} \vec{\nabla} \cdot \vec{V} &= 0, \\ \frac{\partial \vec{\Omega}}{\partial t} &= \vec{\nabla} \times (\vec{V} \times \vec{\Omega}). \end{aligned}$$

   [1+3+2]

2. **Isothermal accretion:**

   (i) Show that the equations governing the spherical accretion for the isothermal case can be written in dimensionless form as

   $$x^2 y z = \lambda, \quad \frac{z^2}{2} + \ln y - \frac{1}{x} = 0,$$

   where

   $$x = \frac{r c_s^2(\infty)}{GM}, \quad y = \frac{\rho}{\rho(\infty)}, \quad z = \frac{|V_r|}{c_s(\infty)},$$

   and λ is a constant.

   (ii) Manipulate these equations to give

   $$\left(z - \frac{1}{z}\right) \mathrm{d}z = \left(\frac{2}{x} - \frac{1}{x^2}\right) \mathrm{d}x.$$

   (iii) Show that the sonic transition occurs when λ has a critical value $\lambda_c = \mathrm{e}^{3/2}/4 \approx 1.12$.

   [5+2+2]

3. **Analytical solutions of the Lane-Emden equation:** Solve the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n; \qquad \theta(0) = 1, \qquad \left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi=0} = 0$$

for $n = 0$ and $n = 1$. Find the form of $\theta(\xi)$ and the value of ξ_1 (defined so that $\theta(\xi_1) = 0$) in each case.

[10]

4. **Mean molecular weight:** Suppose a gas is completely ionized (as is expected within stellar interiors with very high temperatures) and consists of hydrogen (with mass fraction X), helium (with mass fraction Y) and other heavier metals (with mass fraction Z). Obviously $X + Y + Z = 1$. Show that the mean molecular weight in this case is

$$\mu = \left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)^{-1}.$$

You can assume that a heavier element with atomic number A has A neutrons and that $A \gg 1$.

[5]