Statistical Mechanics : Mid-Semester Examination

22nd September, 2003

• Total Marks - 50 • Total Time - 2 Hours •

Thermodynamics

1. If a gas has an equation of state of the form

$$PV = \Phi(T), \quad \text{Boyle's Law}, \tag{1}$$

with internal energy U a function of T alone, i.e,

$$U = \psi(T), \quad \text{Gay-Lussac's and/or Joule's Law}, \tag{2}$$

then deduce from the first and second laws of thermodynamics that the gas is an ideal gas, i.e, has an equation of state given by,

$$PV = RT, \tag{3}$$

where R is a constant. [5]

2. A process where application of pressure brings about a change of temperature is called *throttling*. It takes place at constant *enthalpy*, H. Obtain an expression for the *throttling coefficient*, $J = \partial_P T|_H$, in terms of C_P, V, T, α. (Cooling by the throttling process, also known as the *Joule-Thompson Effect* has very wide engineering applications.) [5]

Ensemble Formulation

1. Using the canonical ensemble formalism, show that,

$$< \mathcal{H}^2 > - < \mathcal{H} >^2 = kT^2 C_V, \tag{4}$$

where $\mathcal{H}$ is the Hamiltonian of a given thermodynamic system. From this result deduce that the specific heat C_V at constant volume is non-negative. [5]

2. Consider a statistical description of thermodynamic systems where E, N, V are allowed to vary. Let us call this a *Super-Canonical Ensemble* approach. Write down the formal expression for, $\mathcal{Z}_{\mathcal{S}}$, the *super-partition function*. Find $< V >$ and $< (\Delta V)^2 >$ as appropriate derivatives of $\mathcal{Z}_{\mathcal{S}}$. [10]

Non-Interacting Systems

1. Show that the partition function $Q(N, V, T)$ of a gas of N ultra-relativistic particles, with energy momentum relation given by $\epsilon = cp$, is

$$Q(N, V, T) = \frac{1}{N!}\left(8\pi V \left(\frac{kT}{hc}\right)^3\right)^N, \tag{5}$$

and the average energy is $< E >= 3NkT$. (You can think of this as an example where the equipartition of energy does not work.) [5]

2. Prove that the isothermal magnetic susceptibility, $\chi_T = \partial_H M|_T$, is related to the fluctuations of the magnetization through the following relation,

$$\chi_T = \frac{(\Delta M)^2}{NkT}. \tag{6}$$

Evidently, this has the form of Curie's law, $\chi_T = C/T$. Determine C. [5]

3. The vibration of a diatomic molecule exhibits anharmonicity when the amplitude becomes very large. The energy levels can be given approximately by

$$E_n = \left(n + \frac{1}{2}\right) h\nu - x_e \left(n + \frac{1}{2}\right)^2 h\nu \tag{7}$$

where x_e is a parameter used to represent the degree of anharmonicity. Calculate the effect of anharmonicity on the vibrational specific heat up to first order in x_e. [10]

Interacting Systems

1. A phase transition occurs at $T = 0$ for $1 - D$ Ising Model. Give physical arguements as to why this should be the case. [5]