General Theory of Relativity : Tutorials

Lecturer : Tirthankar Roy Choudhury
Tutors : Tapomoy Guha Sarkar & Sushan Konar

Preparatory School, SERC-THEP
Tezpur University, July 2013

- The conventions followed will be based on ***Classical Theory of Fields*** by Landau & Lifshitz.
- The signature of the metric would be (1, -1, -1, -1).
- We shall use Latin letters $i, j, k, ..$ for four-dimensional indices, taking on values 0,1,2,3. The Greek letters $\alpha, \beta, ..$ would imply summation over the space indices x, y, z.
- Usually we will use the units in which $c = 1$.

I. :

1. Using the space-time diagram discuss the concepts of simultaneity, length contraction and time-dilation.

$$x' = -t \sinh\phi + x \cosh\phi \tag{1}$$
$$t' = t \cosh\phi - x \sinh\phi \tag{2}$$

2. Show that proper time is longest along the straight world-line in STR. (twin paradox)

3. Some practice with the summation convention:

   (a) Is the following statement true: $a_{ij}x^j = a_{kj}x^j$?

   (b) Expand $a_{ij}x^i y^j$.

   (c) Let $Q = b_{ij}y^i x^j$. Substitute $y^i = a^i_j x^j$ into it and write the result. Discuss that an expression like $Q = b_{ij}a^i_j x^j x^j$ is absurd!

   (d) Show the following:

$$\begin{aligned} a_{ij}(x^i + y^i) &= a_{ij}x^i + a_{ij}y^i \\ a_{ij}x^i y^j &\neq a_{ij}y^i x^j \end{aligned}$$

   (e) Show that $a_{ij}x^i x^j = a_{ji}x^i x^j$.

   (f) Write the following expression in a compact form using summation convention: $a_{11}b^{11} + a_{21}b^{12} + a_{31}b^{13} + a_{41}b^{14}$.

   (g) Show that

$$\frac{\partial}{\partial x^k}(a_{ij}x^i x^j) = (a_{ki} + a_{ik})x^i. \tag{3}$$

4. Show that a vector orthogonal to a time-like vector must be space-like.

5. Show that $A^{ij}S_{ij} = 0$ where A^{ij} is a symmetric and S_{ij} is an anti-symmetric tensor.

6. How many independent components are there in an antisymmetric tensor in D dimensions?
7. Show that $\Lambda^T \eta \Lambda = \eta$.
8. There are two particles of mass m_1 and m_2 moving with velocities u_1 and u_2. Show that the effective mass is not $m_1 + m_2$.

II. :

1. A group of N particles is seen to occupy a volume of $dV = dx\, dy\, dz\, dp_x\, dp_y\, dp_z$ in the phase space, so that the number density of particles Π in the phase space is given by: $N = \Pi\, dV$. Show that is invariant under Lorentz transformations.
2. Make a coordinate transformation to the flat space time described by cylindrical polar coordinates to go to an uniformly rotating frame.
   - Write down the new metric.
   - Identify the non-inertial forces.
   - Show that in the new frame a full rotation gives an excess to the circumference.
   - Show that it is it not possible to synchronize clocks everywhere in such a geometry.
3. If A_i is a covariant vector and $C^{ij} A_i A_j$ is an invariant, show that $C^{ij} + C^{ji}$ is a contravariant tensor of rank 2.
4. How does the Kronecker delta δ^i_k transform? Show that it is a mixed tensor.
5. Show that the Levi-Civita symbol ϵ^{ijkl} is not a tensor. How do you make it a tensor?

III. :

1. In the Rindler space $ds^2 = (1 + g\xi)^2 d\eta^2 - d\xi^2$.
   (a) Calculate g^{ij}, g_{ij} and Christoffel symbols.
   (b) Write down the geodesic equations.
   (c) In the equations $d^2t/ds^2 = 0, d^2x/ds^2 = 0$ make a variable substitution to η, ξ co-ordinates. Check that the corresponding equations are equivalent to geodesic equations.
   (d) Make the non-relativistic limit ($c \to \infty$) and prove, that the geodesic equation reduces to the Newtonian non-relativistic equation $d^2\xi/d\eta^2 = -g$.
2. Consider parabolic coordinates p, q related to the ordinary Cartesian coordinates x, y as
$$p(x, y) = x,\;\; q(x, y) = y - cx^2. \tag{4}$$
   Write the flat Euclidean metric $diag(1, 1)$ in parabolic system. If a vector has components $A^p = 1$ and $A^q = 0$ find A^x and A^y .
3. Calculate all the Christoffel symbols for the following metric and find the geodesics.
$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{5}$$

4. Show that

$$\Gamma^i_{ki} = \frac{\partial \ln \sqrt{-g}}{\partial x^k}. \tag{6}$$

Also show that

$$A^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} A^i)}{\partial x^i}. \tag{7}$$

IV. :

1. In non-relativistic mechanics the Lagrangian of a particle in a gravitational field is given by, Work out the metric in the limit of weak gravitational field.
2. For the metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2) \tag{8}$$

obtain the geodesic equations and show that the motion of the particle is 1-dimensional in an effective potential of the form,

$$V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}. \tag{9}$$

V. :

1. How many independent components are there of the Riemann curvature tensor in D-space-time dimensions?
2. Show that,

$$A^i_{;kl} - A^i_{;lk} = -A^m R^i{}_{mkl}. \tag{10}$$

3. Show that

$$R^{il}_{;i} = \frac{1}{2}(g^{il} R)_{;i}. \tag{11}$$

4. Calculate the independent components of Riemann tensor for the metric on a 2-sphere :

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2. \tag{12}$$

VI. :

1. For the metric

$$ds^2 = dt^2 - a^2(t) dr^2, \tag{13}$$

a. Derive the Einstein tensor $G^{ij} = R^{ij} - \frac{1}{2} g^{ij} R$.
b. For the energy momentum tensor given by $T^{ij} = diag(\rho, p, p, p)$ write down the conservation equation for this metric.
c. Write down the geodesic equation and integrate it.
d. Show that the magnitude of the momentum 3-vector is $\propto 1/a$.
e. Write down the equations of geodesic deviation.

2. On the surface of a sphere, show that, along the geodesic $\phi =$ constant, the geodesic deviation vector ξ' satisfies,

$$\frac{D^2\xi^\theta}{Ds^2} = 0, \ \frac{D^2\xi^\phi}{Ds^2} = -\xi^\phi (d\theta/ds)^2. \tag{14}$$

VII. :

1. Prove the Bianchi identity given by,

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0. \tag{15}$$

2. Starting from the action $\int R\sqrt{-g}d^4x$ obtain the Einstein field equations.
3. Consider a spherically symmetric, static metric of the form,

$$ds^2 = e^{\nu(r)}d\tau^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{16}$$

in presence of matter given with stress-tensor given by,

$$T^0_0 = \rho, T^1_1 = T^2_2 = T^3_3 = -P. \tag{17}$$

Find dP/dr.

VIII. :

1. Consider the metric in the weak field limit

$$g_{ik} = \eta_{ik} + h_{ik}, \quad |h_{ik}| \ll 1, \tag{18}$$

where h_{ik} are small corrections to the Minkowski metric.

(a) Show that to the lowest order in h_{ik}, the contravariant components are given by

$$g^{ik} = \eta^{ik} - h^{ik}, \tag{19}$$

where $h^{ik} = \eta^{im}\eta^{kn}h_{mn}$.

(b) If the metric tensor has the form $\eta_{ik} + h_{ik}$ in a chosen coordinate system, it is always possible to find another coordinate system x'^i where the metric has a similar form. Let us take a infinitesimal coordinate transformation of the form

$$x'^i = x^i + \xi^i, \tag{20}$$

where ξ^i are four arbitrary functions of x^k of the same order of smallness as h_{ik}. Show that in the new coordinate system

$$h'_{ik} = h_{ik} - \frac{\partial \xi_k}{\partial x^i} - \frac{\partial \xi_i}{\partial x^k}. \tag{21}$$

This type of coordinate transformations are known as **gauge transformation**.

(c) Show that the Ricci tensor to the lowest order is given by

$$R_{li} = \frac{1}{2}\left(\frac{\partial^2 h^k_i}{\partial x^k \partial x^l} - \frac{\partial^2 h}{\partial x^i \partial x^l} + \Box h_{li} + \frac{\partial^2 h^n_l}{\partial x^i \partial x^n}\right), \tag{22}$$

where we have defined the D'Alembertian for the Minkowski metric as

$$\Box \equiv -\eta^{ik}\frac{\partial^2}{\partial x^i \partial x^k} = \nabla^2 - \frac{\partial^2}{\partial t^2}. \tag{23}$$

Calculate the Ricci scalar R and the Einstein tensor G_{ik}.

(d) Let us define a new tensor

$$ \tag{24} $$

which is known as "trace reverse" of h_{ik}. Show that the Einstein tensor then becomes

$$G_{ik} = \frac{1}{2}\left(\frac{\partial^2 \bar{h}_i^m}{\partial x^m \partial x^k} + \frac{\partial^2 \bar{h}_k^n}{\partial x^i \partial x^n} + \Box \bar{h}_{ik} - \eta_{ik}\frac{\partial^2 \bar{h}^{mn}}{\partial x^m \partial x^n}.\right) \tag{25}$$

(e) Now use a suitable gauge transformation to reduce the Einstein tensor to a form

$$G_{ik} = \frac{1}{2}\Box \bar{h}_{ik}. \tag{26}$$

What are the conditions on $\bar{h}_{ik}$?

(f) Hence show that in vacuum, the weak field solutions satisfy the standard wave equation. These are known as **gravitational waves**.

IX. : Assume the following Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2) \tag{27}$$

for all the problems below.

1. Find the radius of the smallest stable circular orbit.
2. Verify Kepler's law $T^2 \propto R^3$.
3. A satellite of mass m_1 is in a stable orbit at radius r_1. It ejects a small mass m_2 which moves radially by a distance of 1.1 KM and goes into a stable orbit at r_2. If $r_1 = 100\ KM$ and $r_2 = 99\ KM$ then find the Schwarzschild radius of the gravitating object.
4. For a point source and a point-like gravitational lens derive the lens equations using the formula for bending of light.
5. Write down the metric in the Eddington-Finkelstein coordinates defined by $(t, r, \theta, \phi) \rightarrow (v, r, \theta, \phi)$,

$$t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|, \tag{28}$$

and discuss the nature of the light-cones.

6. Write the modified form of the metric using the following coordinate transformations,

$$\text{for}\, r > 2M,\ U = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \tag{29}$$

$$V = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \tag{30}$$

$$\text{for}\, r < 2M,\ U = \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right) \tag{31}$$

$$V = \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right) \tag{32}$$

Study the causal structure in the $U - V$ plane.

X :

1. Derive the form of the spatial part of FRW metric by embedding a 3-D surface of constant curvature in 4-D flat space.
2. For a cosmological model where the universe is only filled with matter with equation of state $P = w\rho$, calculate the form of the scale factor as a function of time.
3. Calculate the comoving distance r and the age of the universe for the following models,
   - $\Omega_m = 1$ and all other Ω s are equal to zero.
   - $\Omega_m + \Omega_k = 1$ and all other Ω s equal to zero.
   - $\Omega_\lambda = 1$, all other Ω s are equal to zero.

XI :

1. Consider a matter component having a time-varying equation of state, i.e., $P(a) = w(a)\rho(a)$. Find the evolution of ρ as a function of a.
2. (a) Show that the comoving distance to a galaxy at redshift z can be expanded in a power series as

$$d_C(z) \equiv R_0 S_k^{-1}(r) = H_0^{-1}\left[z - \frac{1}{2}(1+q_0)z^2 + \dots\right], \tag{33}$$

   where

$$q_0 \equiv -\frac{\ddot{R}\,R}{\dot{R}^2} \tag{34}$$

   is the deceleration parameter at the present time t_0.

   (b) What is the series expansion for the proper distance $d_P(z) \equiv R(t)S_k^{-1}(r)$?
3. Let $\phi(t)$ be a homogeneous scalar field (i.e., independent of the spatial coordinates) having a Lagrangian

$$L = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{35}$$

   Calculate the stress-energy tensor. Also calculate the pressure and density for this scalar field. Under what condition does the equation of state become equal to -1?