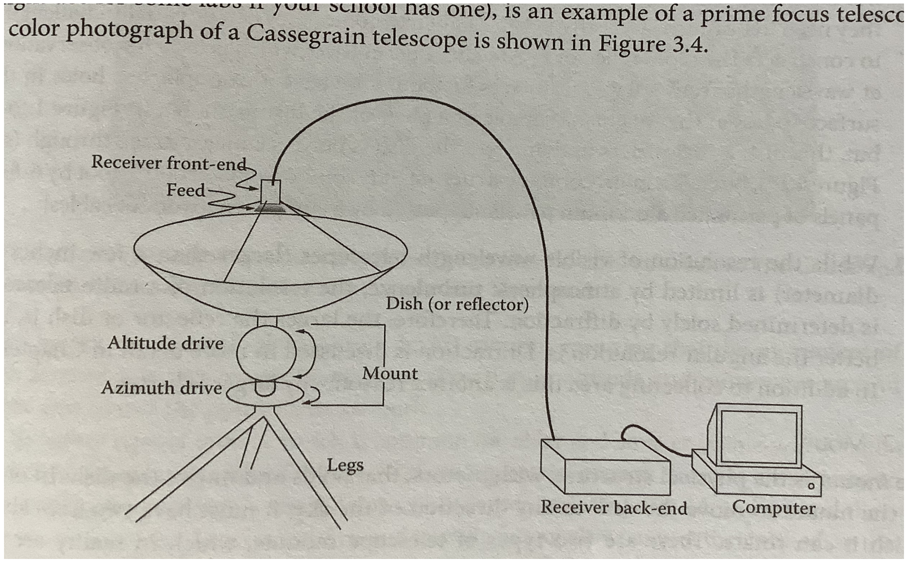
- Recap
- Weiner-Khinchin theorem
- Van Cittert-Zernicke theorem

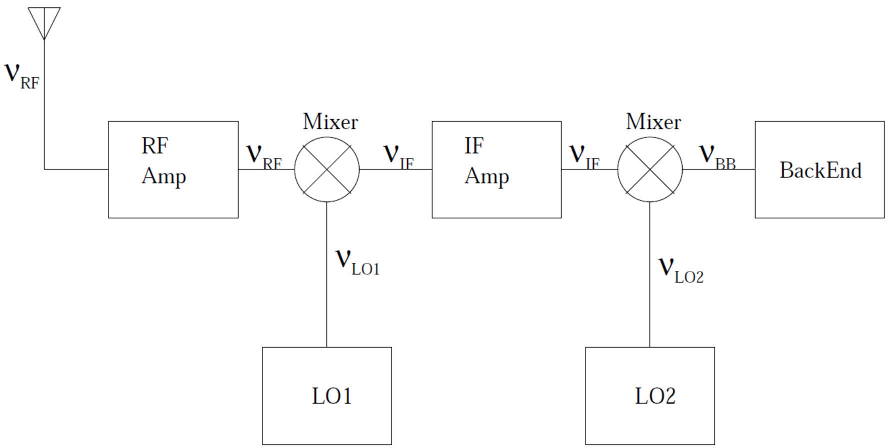
Astronomical Techniques II : **Lecture 3**

Ruta Kale

Low Frequency Radio Astronomy (Chp. 1, 2)
Synthesis imaging in radio astronomy II, Chp 1

A basic radio telescope





-Feed
-Receiver front end
-Reflector
-Mount
-Transmission lines
-Receiver back-end
-Computer

- Brightness temperature, Antenna temperature
- Antenna parameters (directivity, illumination, gain, surface errors)
- Far field antenna pattern = FT(aperture distribution)

The Wiener-Khinchin Theorem

Consider a random process x(t). The auto-correlation of x is defined as

$$r_{xx}(t,\tau) = \langle x(t)x(t+\tau)\rangle$$

for stationary signals

$$r_{xx}(\tau) = \langle x(t)x(t+\tau)\rangle$$

where angular brackets indicate taking the mean value.

The Fourier transform S(ν) of the auto-correlation function is the power spectrum:

$$S(\nu) = \int_{-\infty}^{\infty} r_{xx}(\tau) e^{-i2\pi\tau\nu} d\tau$$

and

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} S(\nu) e^{i2\pi\tau\nu} d\nu$$

The auto-correlation function is the Fourier transform of the power spectrum.
- Weiner-Khinchin theorem

The Wiener-Khinchin Theorem

Example: A process whose auto-correlation function is a delta function has a power spectrum that is flat – “white noise”.

In radio astronomy we usually have *band-limited signals* - in this case auto-correlation is a sinc function with a width ~1/$\Delta\nu$.

This width is also called the “coherence time” of the signal.

$$S(\nu) = \int_{-\infty}^{\infty} r_{xx}(\tau) e^{-i2\pi\tau\nu} d\tau \quad \text{and} \quad r_{xx}(\tau) = \int_{-\infty}^{\infty} S(\nu) e^{i2\pi\tau\nu} d\nu$$

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Temporal and spatial correlations

In the previous example we had random processes that are a function of time alone. But the signal received from a distant cosmic source is in general a function of both time and receiver location. One can also define spatial correlation functions.

Consider the signal *E(r)* at a particular instant in the observer's plane, then the spatial correlation function is:

$$V(x) = \langle E(r)E^*(r+x)\rangle$$

This function V is referred to as the visibility and is central to the topic of interferometry.

Can we do without interferometry ?

Rayleigh's criterion (resolution is diffraction limited) for an aperture size of size D,

$$\theta \sim \lambda / D$$

Resolution of single dishes in radio bands:

For a dish of diameter 10 m and observing frequency of 21cm = ?

To match the resolution of our eye ~20'', at 21cm we need a dish of diameter ~ ?? (calculate).

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Hard to learn about sources in the absence of a match with optically known sources.

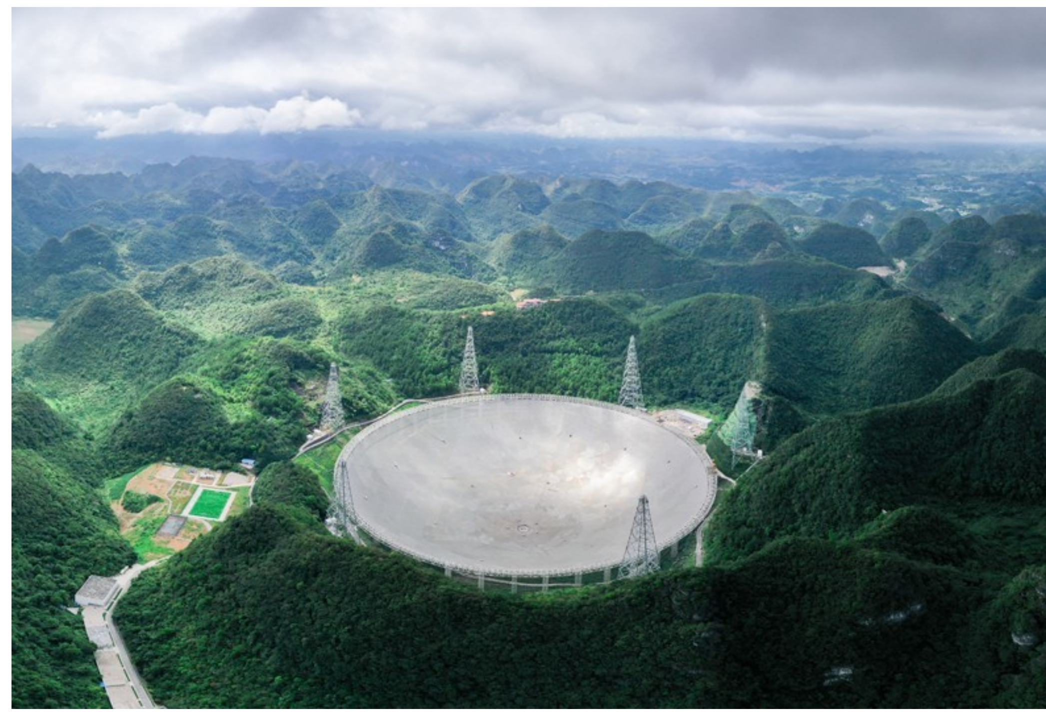
In optical, resolutions are sub-arcsec – limited by atmospheric "seeing". Impractical mechanically to make antennas of such dimensions for radio wavelengths.

Single dish telescopes

However for certain observations single dish telescopes are still useful and are being built.

Arecibo (operational since Nov 1963)
305 m
Collapsed (Dec 1, 2020)

Five hundred metre Aperture Spherical Telescope (FAST), since 2016
China



Van Cittert-Zernicke theorem

This relates the spatial coherence function, $V(r_1,r_2) = <E(r_1)E^*(r_2)>$ to the intensity distribution of the incoming radiation $I(s)$. It shows that $V(r_1,r_2)$ only depends on $r_1 - r_2$ and if all the measurements are in a plane,

$$V(r_1,r_2) = F\{I(s)\}$$

Proof in "Principles of Optics" by Born and Wolf (Chapter 10).

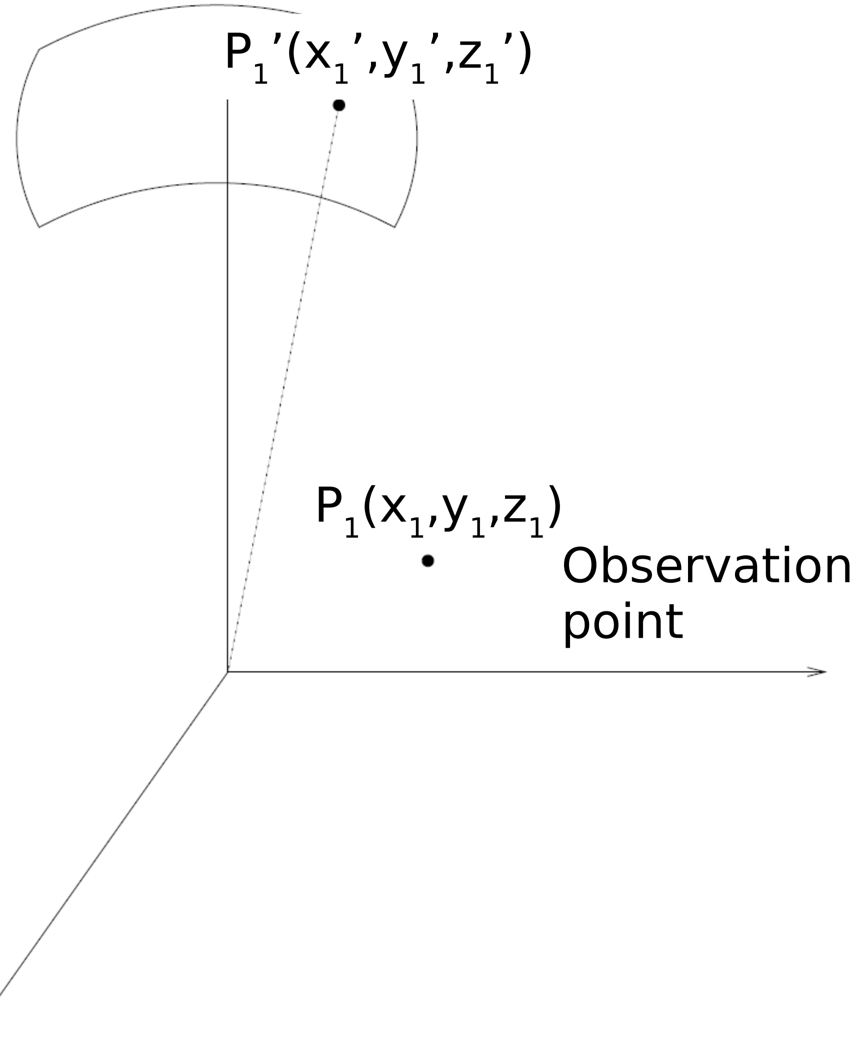
Van Cittert-Zernicke theorem

Consider a *distant* source approximated as a brightness distribution on the celestial sphere located at distance R from the observer. Let the electric field at the point P_1' be $\varepsilon(P_1')$.

The electric field $E(P_1)$ at the observation point can be given by,

$$E(P_1) = \int \varepsilon(P_1') \frac{e^{-ikD(P_1', P_1)}}{D(P_1', P_1)} d\Omega_1$$

$D(P_1', P_1)$ = Distance between P_1 and P_1'



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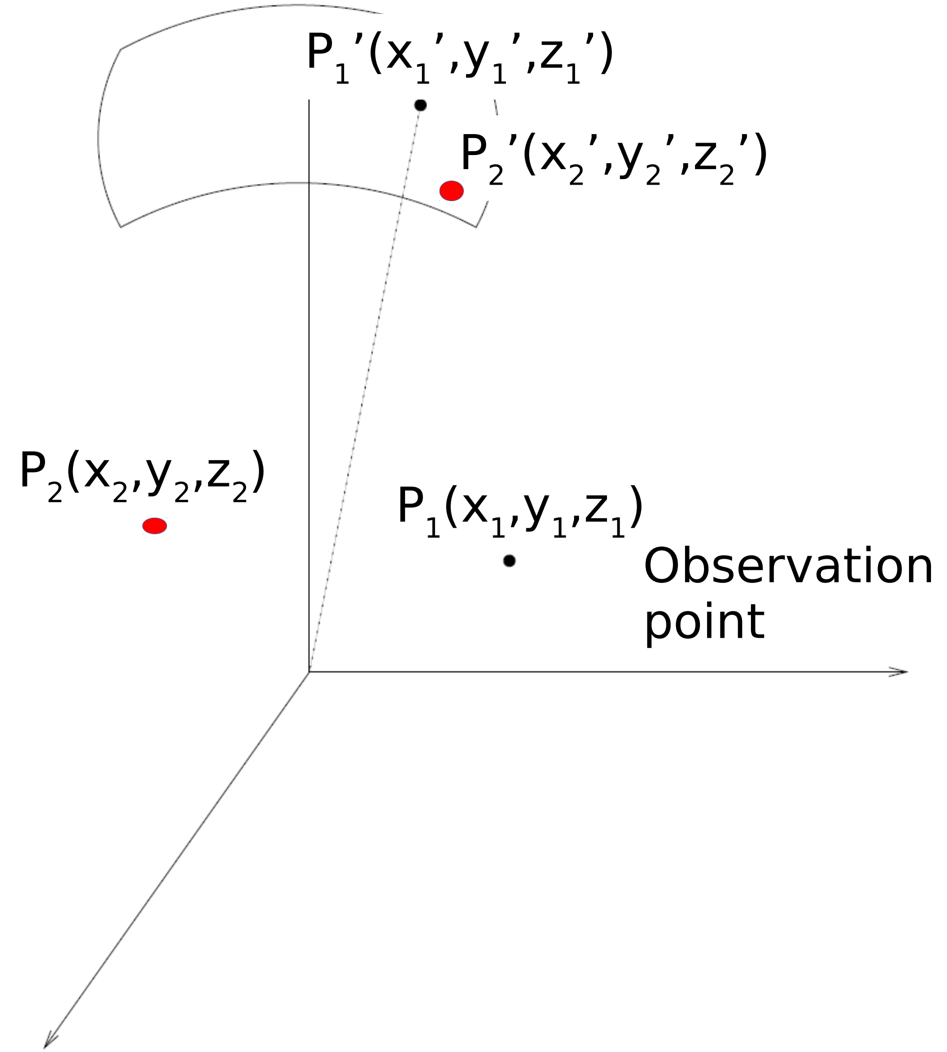
Consider another point P_2 and P_2' and the field at P_2.



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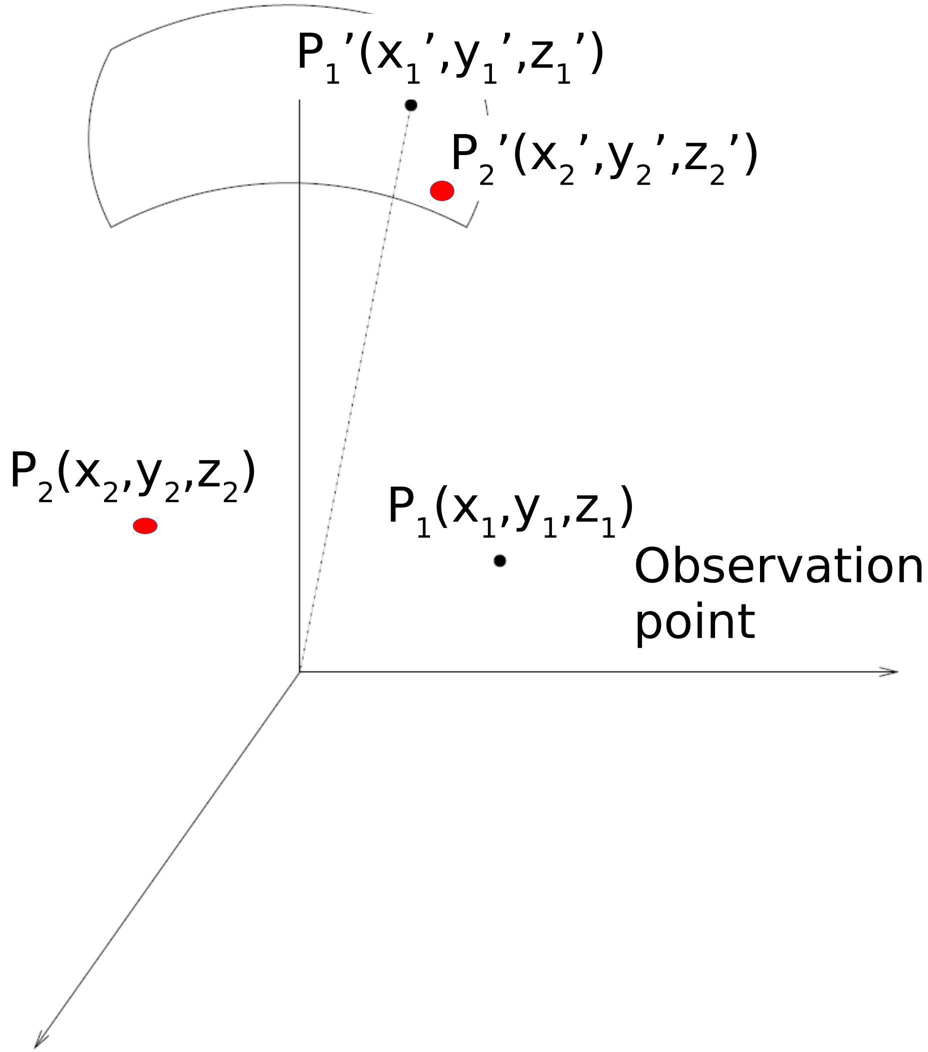
Consider another point P_2 and P_2' and the field at P_2.

Aim is to find the *cross-correlation between the two fields*: $\langle E(P_1) E^*(P_2) \rangle$



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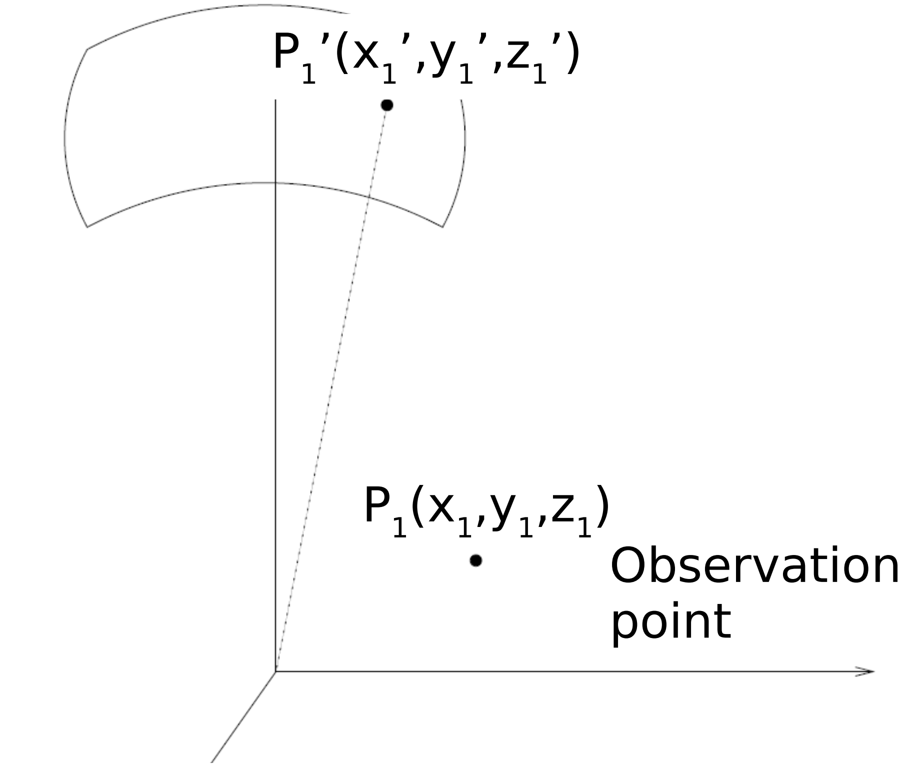

$$\langle E(P_1)E^*(P_2)\rangle = \int \langle \varepsilon(P_1')\varepsilon^*(P_2')\rangle \frac{e^{-ik[D(P_1',P_1)-D(P_2',P_2)]}}{D(P_1',P_1)D(P_2',P_2)} d\Omega_1 d\Omega_2$$

Van Cittert-Zernicke theorem

$$\langle E(P_1)E^*(P_2)\rangle = \int \langle \varepsilon(P_1')\varepsilon^*(P_2')\rangle \frac{e^{-ik[D(P_1',P_1)-D(P_2',P_2)]}}{D(P_1',P_1)D(P_2',P_2)} d\Omega_1 d\Omega_2$$

Assuming that the emission from the source is incoherent then,

$$\langle \varepsilon(P_1')\varepsilon^*(P_2')\rangle = 0$$ except when $$P_1' = P_2'$$

Replace P_2’ with P_1’

<$\varepsilon(P_1')\varepsilon^*(P_1')$> is the intensity I at the point P_1’

Van Cittert-Zernicke theorem

$$\langle E(P_1)E^*(P_2)\rangle = \int \langle \varepsilon(P_1')\varepsilon^*(P_2')\rangle \frac{e^{-ik[D(P_1',P_1)-D(P_2',P_2)]}}{D(P_1',P_1)D(P_2',P_2)} d\Omega_1 d\Omega_2$$

Assuming that the emission from the source is *incoherent* then,

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$$\langle E(P_1)E^*(P_2)\rangle = \int I(P_1') \frac{e^{-ik[D(P_1',P_1)-D(P_1',P_2)]}}{D(P_1',P_1)D(P_1',P_2)} d\Omega_1$$

Van Cittert-Zernicke theorem

$$D(P_1', P_1) = [(x_1' - x_1)^2 + (y_1' - y_1)^2 + (z_1' - z_1)^2]^{1/2}$$

$$x_1' = Rcos(\theta_x) = Rl$$
$$y_1' = Rcos(\theta_y) = Rm$$
$$y_1' = Rcos(\theta_z) = Rn$$

$$l^2 + m^2 + n^2 = 1$$

$$d\Omega = \frac{dl\ dm}{\sqrt{1 - l^2 - m^2}}$$

Derive the following approximation:

$$D(P_1', P_1) \simeq R - (lx_1 + my_1 + nz_1)$$

Similarly for $D(P_1', P_2)$

Van Cittert-Zernicke theorem

Substituting in the equation:

$$\langle E(P_1)E^*(P_2)\rangle = \int I(P_1')\frac{e^{-ik[D(P_1',P_1)-D(P_1',P_2)]}}{D(P_1',P_1)D(P_1',P_2)}d\Omega_1$$

$$\langle E(P_1)E^*(P_2)\rangle = \int I(l,m)e^{-ik[l(x_2-x_1)+m(y_2-y_1)+n(z_1-z_1)]}\frac{dldm}{\sqrt{1-l^2-m^2}}$$

Notice *I* is now written as a function of l and m: only two direction cosines are sufficient to uniquely specify a position on the celestial sphere.
We have also dropped the constant R^2 from the denominator.

Further we express the coordinates in units of wavelength.

Van Cittert-Zernicke theorem

$$\langle E(P_1)E^*(P_2)\rangle = \int I(l,m)e^{-ik[l(x_2-x_1)+m(y_2-y_1)+n(z_1-z_1)]}\frac{dldm}{\sqrt{1-l^2-m^2}}$$

$$u = (x_2 - x_1)/\lambda$$
$$v = (y_2 - y_1)/\lambda$$
$$w = (z_2 - z_1)/\lambda$$

$$V(u,v,w) = \int I(l,m)e^{-i2\pi[lu+mv+nw]}\frac{dldm}{\sqrt{1-l^2-m^2}}$$

Looks like a Fourier transform.

Spatial correlation of the electric field is related to the source brightness distribution.

Special cases

Observations are confined to the u-v plane, w = 0:

$$V(u,v) = \int \frac{I(l,m)}{\sqrt{1-l^2-m^2}} e^{-i2\pi[lu+mv]} dl dm$$

Source brightness is limited to a small region of the sky -

$n = \sqrt{1-l^2-m^2} \simeq 1$

$$V(u,v,w) = e^{-i2\pi w} \int I(l,m) e^{-i2\pi[lu+mv]} dl dm$$