- A two element interferometer

Astronomical Techniques II : **Lecture 4**

Ruta Kale

Low Frequency Radio Astronomy (Chp. 4)
http://www.ncra.tifr.res.in/ncra/gmrt/gmrt-users/low-frequency-radio-astronomy

Synthesis imaging in radio astronomy II, Chp 2

Interferometry and synthesis in radio astronomy (Chp 2)

The Wiener-Khinchin Theorem

Consider a random process x(t). The auto-correlation of x is defined as

$$r_{xx}(t,\tau) = \langle x(t)x(t+\tau)\rangle$$

for stationary signals

$$r_{xx}(\tau) = \langle x(t)x(t+\tau)\rangle$$

where angular brackets indicate taking the mean value.

The Fourier transform S(ν) of the auto-correlation function is the power spectrum:

$$S(\nu) = \int_{-\infty}^{\infty} r_{xx}(\tau)e^{-i2\pi\tau\nu}d\tau$$

and

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} S(\nu)e^{i2\pi\tau\nu}d\nu$$

The power spectrum of a signal is the Fourier transform of the autocorrelation function of that signal.
- Wiener-Khinchin theorem (or Wienier-Khinchin relation)

Relation between autocorrelation and power spectrum

$$
\begin{array}{ccc}
x_j & \Leftrightarrow & X_k \\
\text{(function)} & \text{DFT} & \text{(transform)} \\
\Downarrow & & \Downarrow \\
x_j \star x_j & \Leftrightarrow & |\,X_k\,|^2 \\
\text{(autocorrelation)} & \text{DFT} & \text{(power spectrum).}
\end{array}
$$

Temporal and spatial correlations

In the previous example we had random processes that are a function of time alone. But the signal received from a distant cosmic source is in general a function of both time and receiver location. One can also define spatial correlation functions.

Consider the signal *E(r)* at a particular instant in the observer's plane, then the spatial correlation function is:

$$V(x) = \langle E(r) E^*(r + x) \rangle$$

Complex conjugate

This function V is referred to as the visibility and is central to the topic of interferometry.

Angular resolution

Rayleigh's criterion (resolution is diffraction limited) for an aperture size of size D,

$$\theta \sim \lambda / D$$

Resolution of single dishes in radio bands:

To match the resolution of our eye ~20'', at 21cm we need a dish of diameter ~ 2 km.

For Grote Reber's single dish of 10 m diameter the resolution for 2m wavelength, ~11 degrees.

For the GMRT single dish (45 m diameter) resolutions for 2m, 1m, 0.5m and 0.21m ????

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Hard to learn about sources in the absence of a match with optically known sources.

Impractical mechanically to make antennas of such dimensions for radio wavelengths.

Single dish telescope examples

Arecibo (operational since Nov 1963)
305 m
Collapsed (Dec 1, 2020)

Five hundred metre Aperture Spherical Telescope (FAST), since 2016 China

Resolution ~ 90” at 21 cm

Interferometry

To achieve high angular resolutions in radio bands “aperture synthesis” was developed and that is based on the concept of “interferometry”.

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Historical Milestones:

- Michelson* stellar interferometer: measurement of fringe amplitude to find angular width of a star (1890-1921)
- Ryle and Vonberg (1946) First two element interferometer
- ~1952 onwards – measurements of angular dimensions of sources by varying baselines
- Tracking antennas (~1960s)
- Earth rotation synthesis – Ryle with some learning from solar experiments done earlier
- Image processing techniques (~1974)
- ...

Interferometry

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There is an analogy between Young’s double slit experiment with quasi-monochromatic light and a two element interferometer – we will go into that after we initiate the concept of interferometry through the Van Cittert Zernicke theorem.

Van Cittert-Zernicke theorem

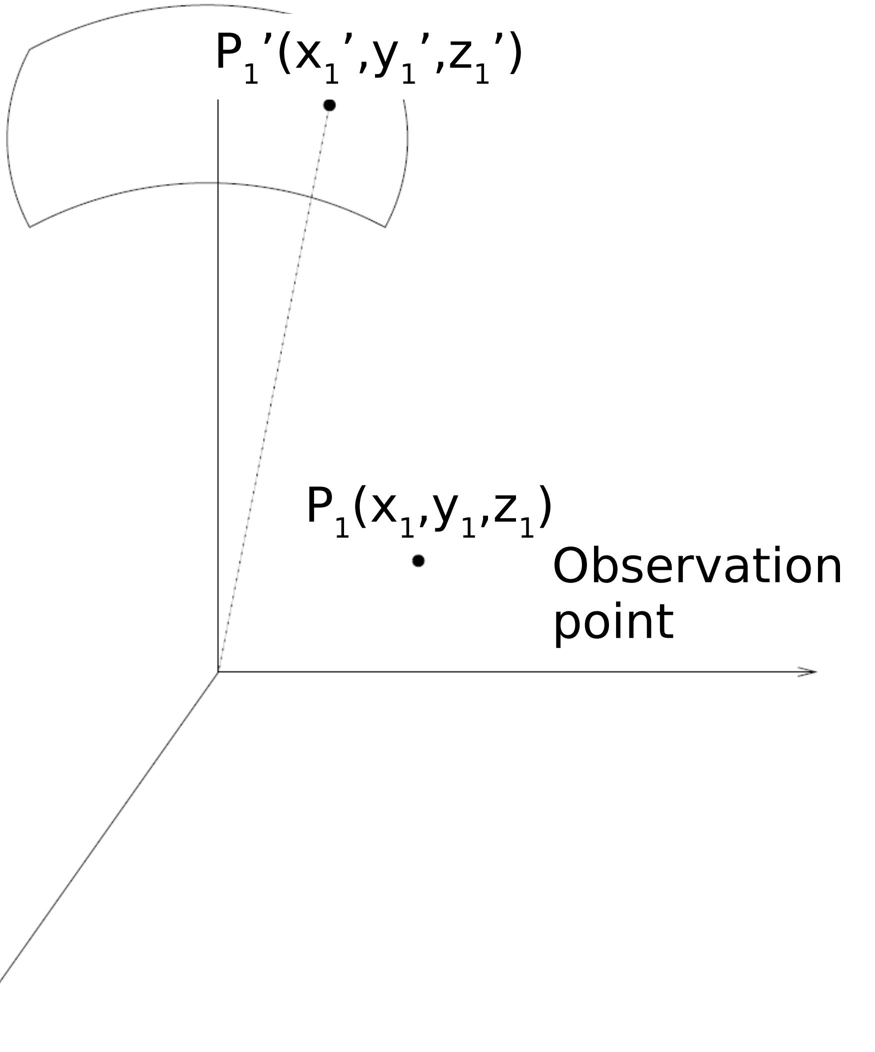
This relates the spatial coherence function, $V(r_1,r_2) = <E(r_1)E^*(r_2)>$ to the intensity distribution of the incoming radiation $I(s)$. It shows that $V(r_1,r_2)$ only depends on $r_1 - r_2$ and if all the measurements are in a plane,

$$V(r_1,r_2) = F\{I(s)\}$$

Proof in "Principles of Optics" by Born and Wolf (Chapter 10).

Van Cittert-Zernicke theorem

Consider a *distant* source approximated as a brightness distribution on the celestial sphere located at distance R from the observer. Let the electric field at the point P_1' be $\mathcal{E}(P_1')$.



$D(P_1', P_1)$ = Distance between P_1 and P_1'

Van Cittert-Zernicke theorem

Consider a *distant* source approximated as a brightness distribution on the celestial sphere located at distance R from the observer. Let the electric field at the point P_1' be $\varepsilon(P_1')$.

The electric field $E(P_1)$ at the observation point can be given by,

$$E(P_1) = \int \varepsilon(P_1') \frac{e^{-ikD(P_1', P_1)}}{D(P_1', P_1)} d\Omega_1$$

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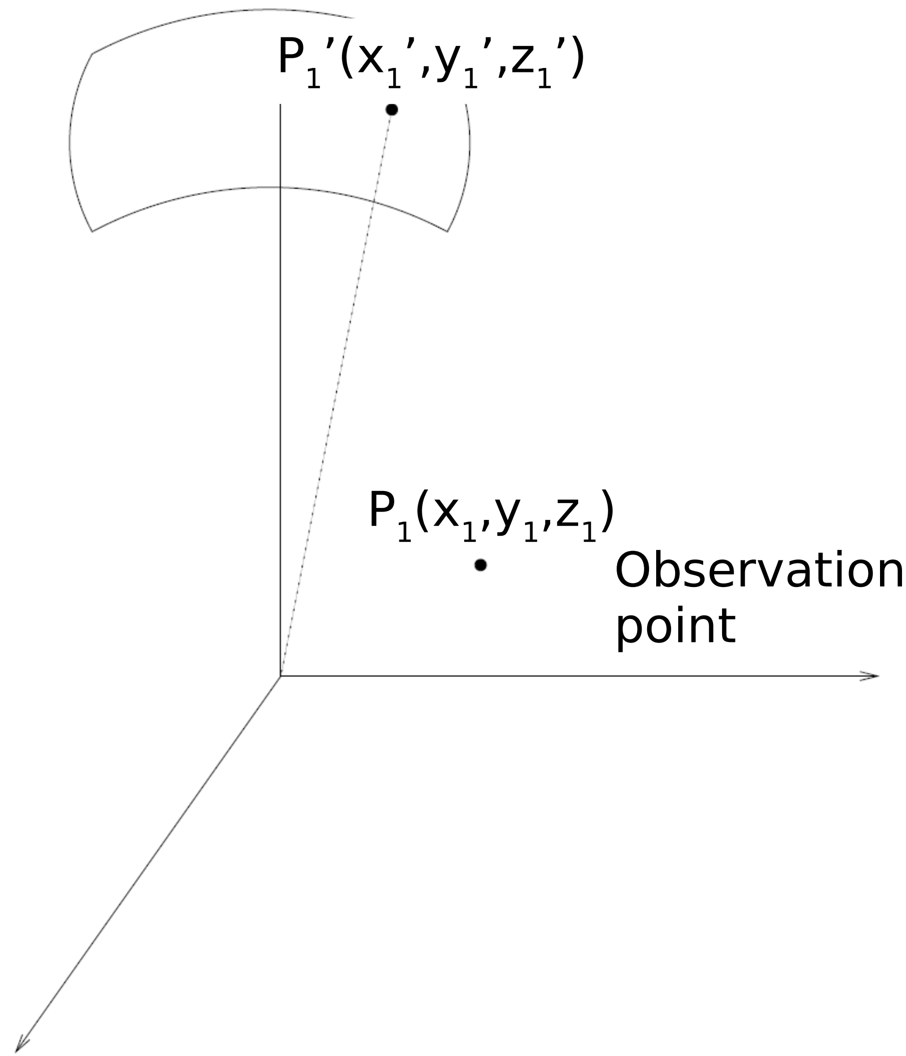

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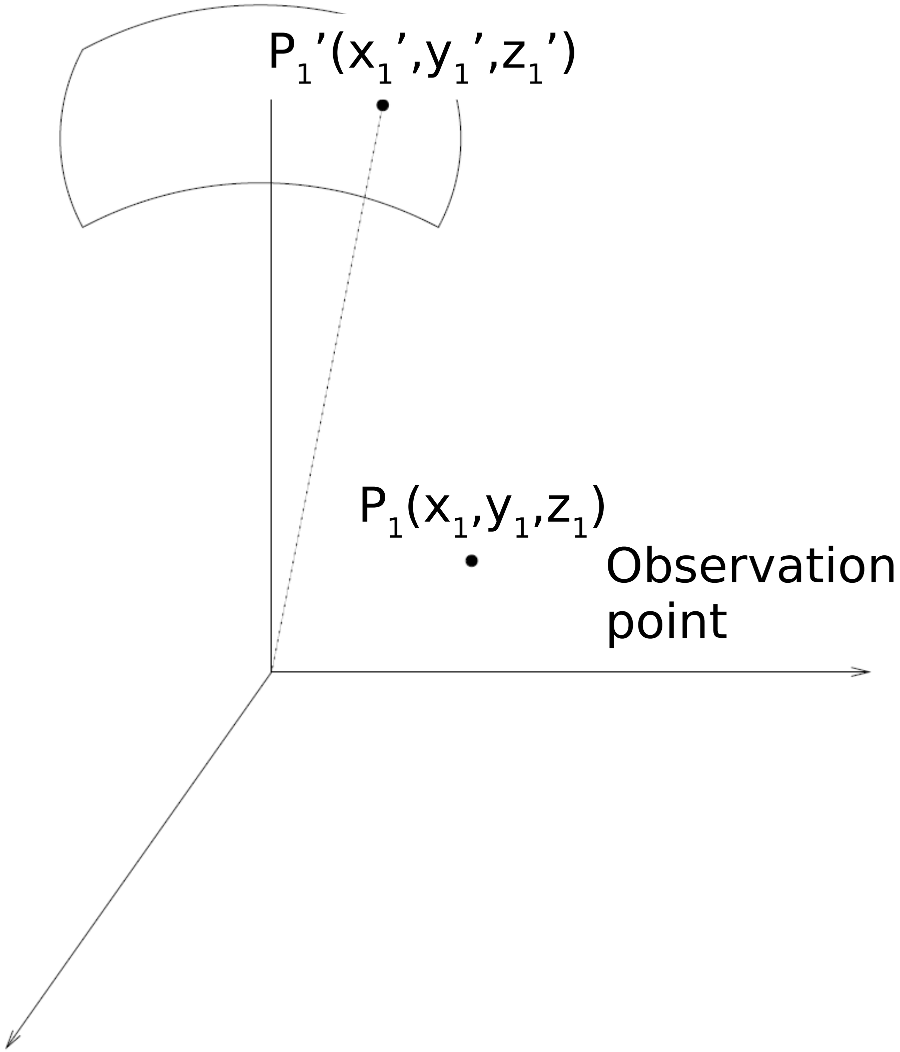

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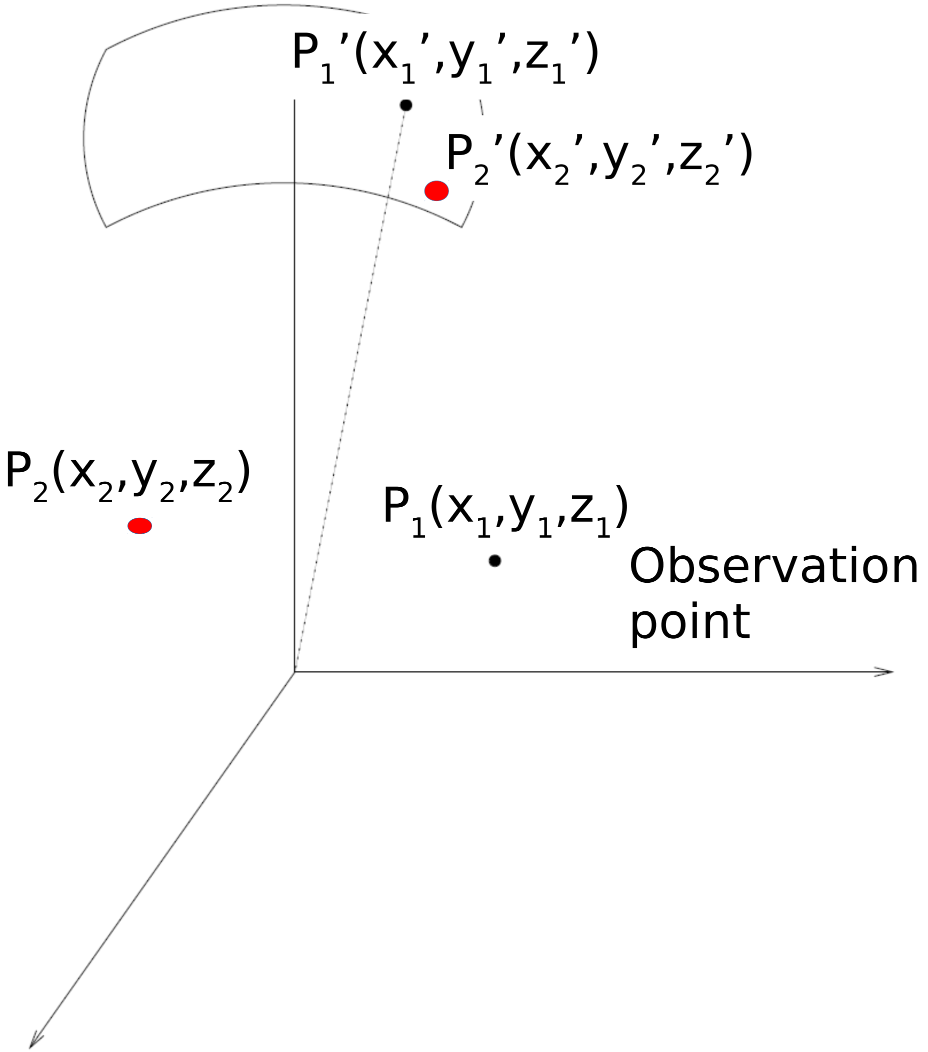
Consider another point P_2 and P_2' and the field at P_2.



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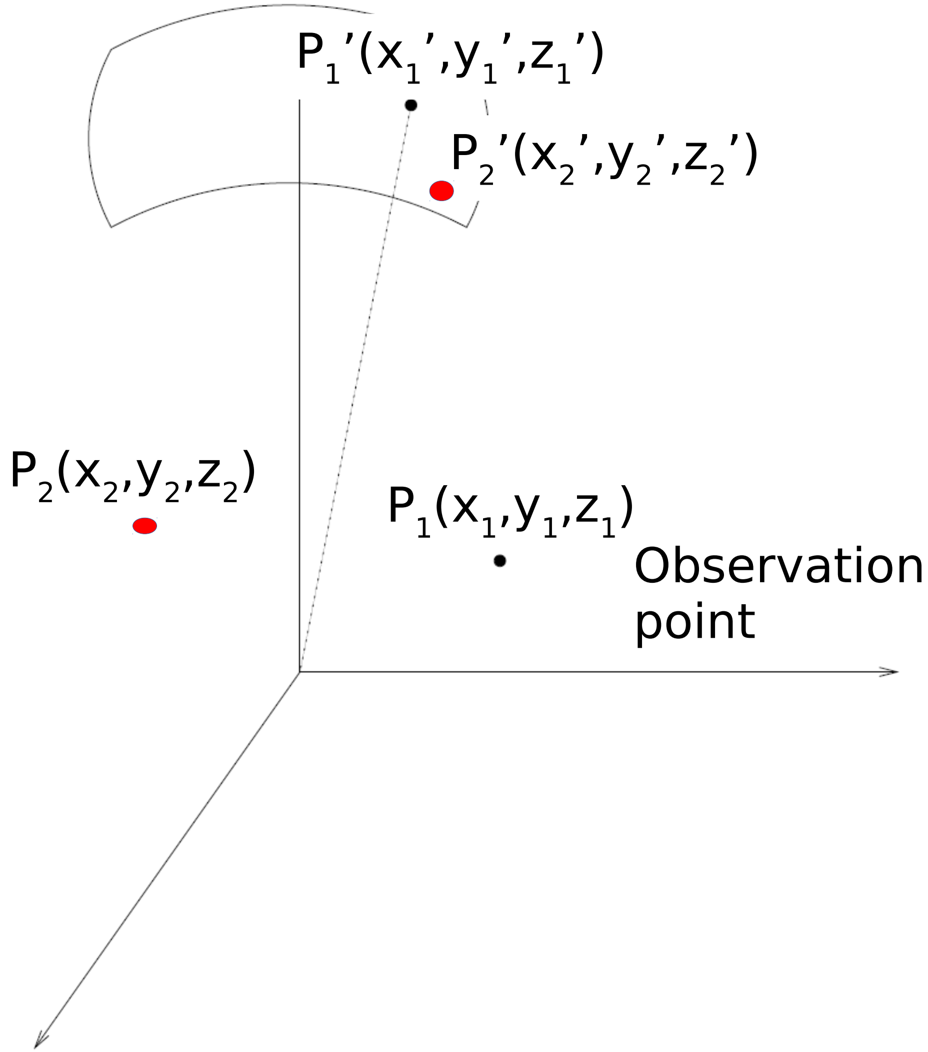

Consider another point P_2 and P_2' and the field at P_2.

Aim is to find the *cross-correlation between the two fields*: $\langle E(P_1)E^*(P_2)\rangle$

Van Cittert-Zernicke theorem

$$\langle E(P_1)E^*(P_2)\rangle = \int \langle \varepsilon(P_1')\varepsilon^*(P_2')\rangle \frac{e^{-ik[D(P_1',P_1)-D(P_2',P_2)]}}{D(P_1',P_1)D(P_2',P_2)} d\Omega_1 d\Omega_2$$

Assuming that the emission from the source is *incoherent* then,

$$\langle \varepsilon(P_1')\varepsilon^*(P_2')\rangle = 0$$ except when $$P_1' = P_2'$$

Replace P_2’ with P_1’

$<\varepsilon(P_1')\varepsilon^*(P_1')>$ is the intensity I at the point P_1’

Van Cittert-Zernicke theorem

$$\langle E(P_1)E^*(P_2)\rangle = \int \langle \varepsilon(P_1')\varepsilon^*(P_2')\rangle \frac{e^{-ik[D(P_1',P_1)-D(P_2',P_2)]}}{D(P_1',P_1)D(P_2',P_2)} d\Omega_1 d\Omega_2$$

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$$\langle E(P_1)E^*(P_2)\rangle = \int I(P_1') \frac{e^{-ik[D(P_1',P_1)-D(P_1',P_2)]}}{D(P_1',P_1)D(P_1',P_2)} d\Omega_1$$

Van Cittert-Zernicke theorem

$$D(P_1', P_1) = [(x_1' - x_1)^2 + (y_1' - y_1)^2 + (z_1' - z_1)^2]^{1/2}$$

$$x_1' = Rcos(\theta_x) = Rl$$
$$y_1' = Rcos(\theta_y) = Rm$$
$$z_1' = Rcos(\theta_z) = Rn$$

Source confined to celestial sphere:

$$l^2 + m^2 + n^2 = 1$$

$$d\Omega = \frac{dl\ dm}{\sqrt{1-l^2-m^2}}$$

Derive the following approximation:

$$D(P_1', P_1) \simeq R - (lx_1 + my_1 + nz_1)$$

Similarly for $D(P_1', P_2)$

Van Cittert-Zernicke theorem

Substituting in the equation:

$$\langle E(P_1)E^*(P_2)\rangle = \int I(P_1')\frac{e^{-ik[D(P_1',P_1)-D(P_1',P_2)]}}{D(P_1',P_1)D(P_1',P_2)}d\Omega_1$$

$$\langle E(P_1)E^*(P_2)\rangle = \int I(l,m)e^{-ik[l(x_2-x_1)+m(y_2-y_1)+n(z_1-z_1)]}\frac{dldm}{\sqrt{1-l^2-m^2}}$$

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Notice *I* is now written as a function of l and m: only two direction cosines are sufficient to uniquely specify a position on the celestial sphere.
We have also dropped the constant R^2 from the denominator.

Further we express the coordinates in units of wavelength.

Van Cittert-Zernicke theorem

$$\langle E(P_1)E^*(P_2)\rangle = \int I(l,m)e^{-ik[l(x_2-x_1)+m(y_2-y_1)+n(z_1-z_1)]}\frac{dldm}{\sqrt{1-l^2-m^2}}$$

$$u = (x_2 - x_1)/\lambda$$
$$v = (y_2 - y_1)/\lambda$$
$$w = (z_2 - z_1)/\lambda$$

$$V(u,v,w) = \int I(l,m)e^{-i2\pi[lu+mv+nw]}\frac{dldm}{\sqrt{1-l^2-m^2}}$$

Looks like a Fourier transform.

Spatial correlation of the electric field is related to the source brightness distribution.

Assumptions

Treated electric field like a scalar (was implicit when we used Huygen's principle).

Sources are far away (assume emission confined to "celestial sphere").

Celestial sphere is empty.

Radiation from astronomical sources is spatially incoherent.

Special cases

Observations are confined to the u-v plane, w = 0:

$$V(u,v) = \int \frac{I(l,m)}{\sqrt{1-l^2-m^2}} e^{-i2\pi[lu+mv]} dldm$$

Source brightness is limited to a small region of the sky -

$n = \sqrt{1-l^2-m^2} \simeq 1$

$$V(u,v,w) = e^{-i2\pi w} \int I(l,m) e^{-i2\pi[lu+mv]} dldm$$