Electrodynamics and Radiative Processes I

Lecture 6 – Relativity in Electrodynamics

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Lecture -6
Questions raised in the class

1. How do we consider the Thomson scattering by unpolarized light?

2. The term terminal velocity is used to denote the constant velocity attained after the forces are balanced. In question number 3 of assignment 1, both the forces are r^2 dependent, hence for acceleration to be zero, the relation between the constants must be such that $\frac{-G_{eff}M}{R} = \frac{1}{2}v^2$ $\quad v^2 = \frac{2}{R}\left(\frac{\kappa L}{4\pi c} - GM\right)$

Compare the same with the terminal velocity in viscous fluid.
@ Sukanya

Super Luminal motion

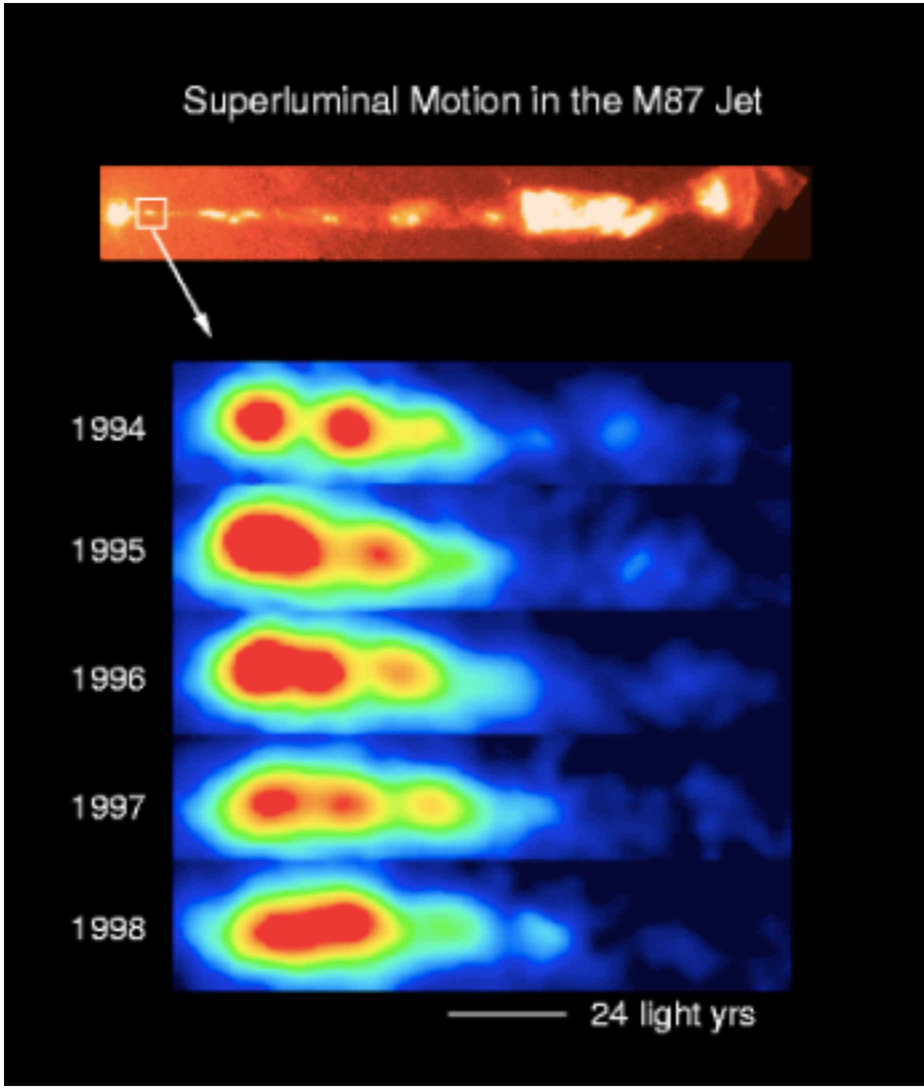
Apparently faster than light motion seen in some radio galaxies, Quasers, blazers etc.
(Problem 4.7 of R&L)

These sources contain a black hole responsible for ejection of mass at high velocities.

$$v_{app}^{max} = \frac{v\sqrt{1-\beta^2}}{1-\beta^2} = \gamma v$$

Large v and γ >>1

Apparent velocities >> c



Picture of Andromeda galaxy,
Photons recorded were emitted up to 200,000 years difference.
You are seeing photons arriving at the same time
NOT emitted at the same time

Review of Lorentz transformation and four vectors

Special Relativity

Special Relativity -- theory describing the motion of particles and fields at any speed.

Based on two principles:

1. All inertial frames are equivalent for all experiments.

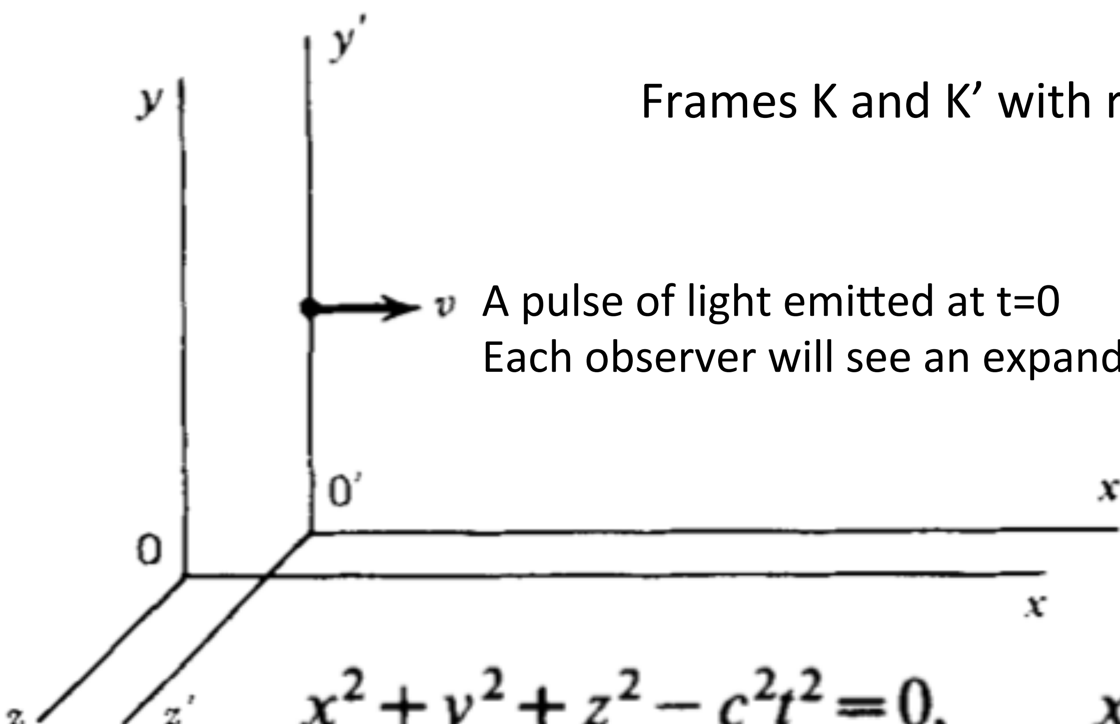
2. Maxwell's equations and the speed of light must be the same for all observers.

Review of Lorentz Transformations

Both space and time are subject to Lorentz transformation.

✓ The laws of nature are the same in two frames of reference in uniform relative motion with no rotation.

✓ The speed of light is c in all such frames



Frames K and K' with relative velocity v along x axis

A pulse of light emitted at t=0

Each observer will see an expanding sphere centered on its origin

Equations of expanding sphere

$$x^2+y^2+z^2-c^2t^2=0, \qquad x'^2+y'^2+z'^2-c^2t'^2=0,$$

Review of Lorentz Transformations

The Andromeda paradox

Formulated first by R. Penrose to illustrate the apparent paradox of relativity of simultaneity

Invisibility of Lorentz transform

https://apatruno.files.wordpress.com/2014/09/terrell.pdf

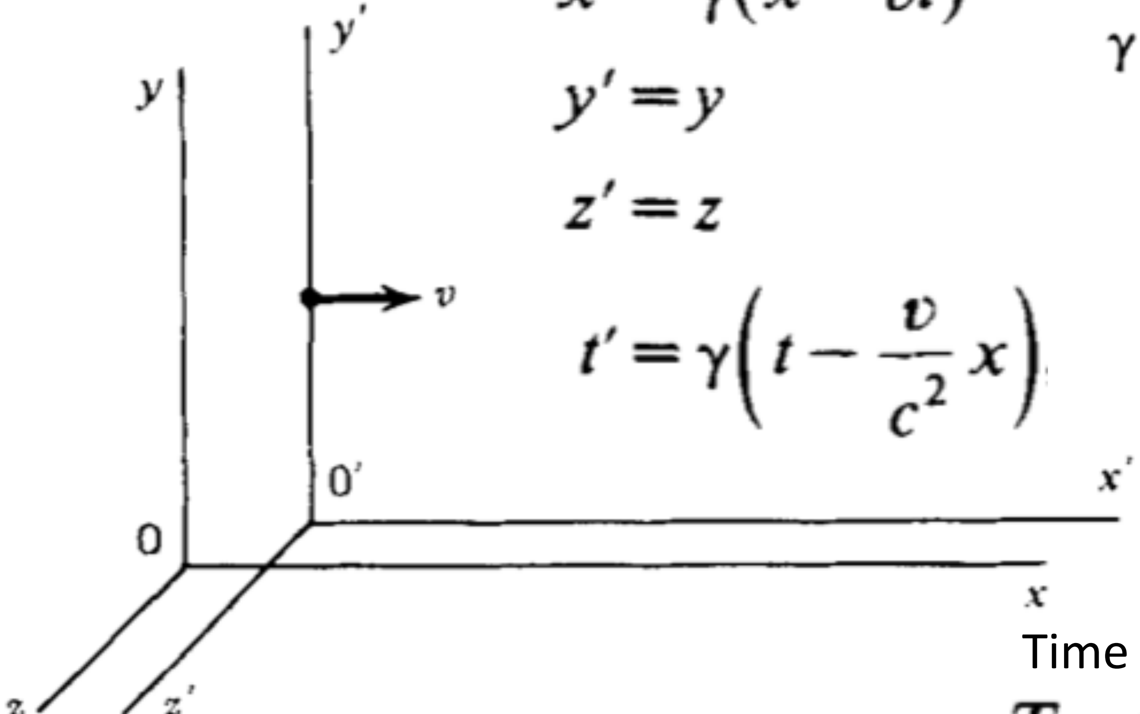
$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$



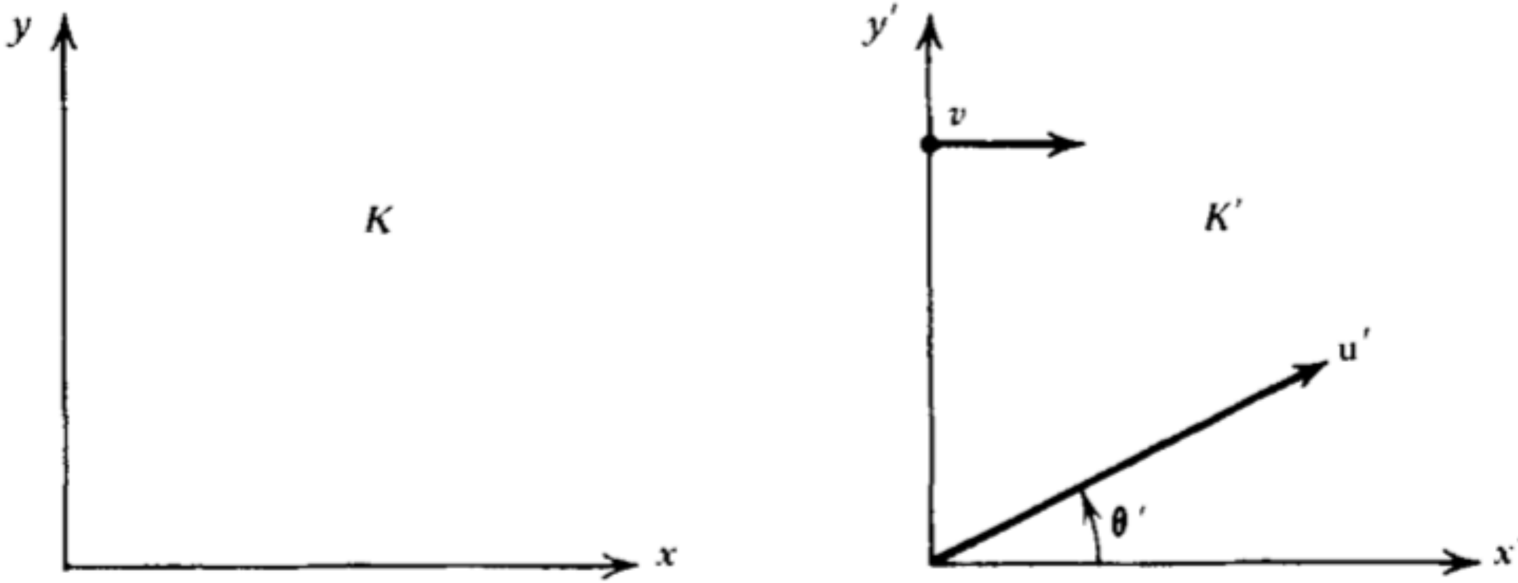
Length contraction

$$L = \left(1 - \frac{v^2}{c^2}\right)^{1/2} L_0$$

Time dilation

$$T = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma T_0$$

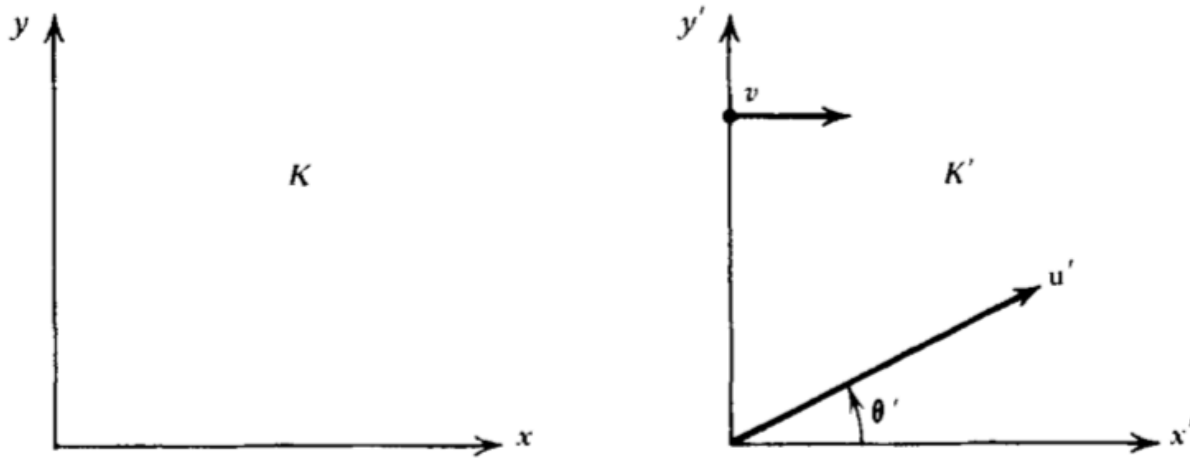
Transformations of velocities



$$dx = \gamma(dx' + v\,dt'), \qquad dy = dy'$$

$$dz = dz', \qquad dt = \gamma\left(dt' + \frac{v}{c^2}\,dx'\right)$$

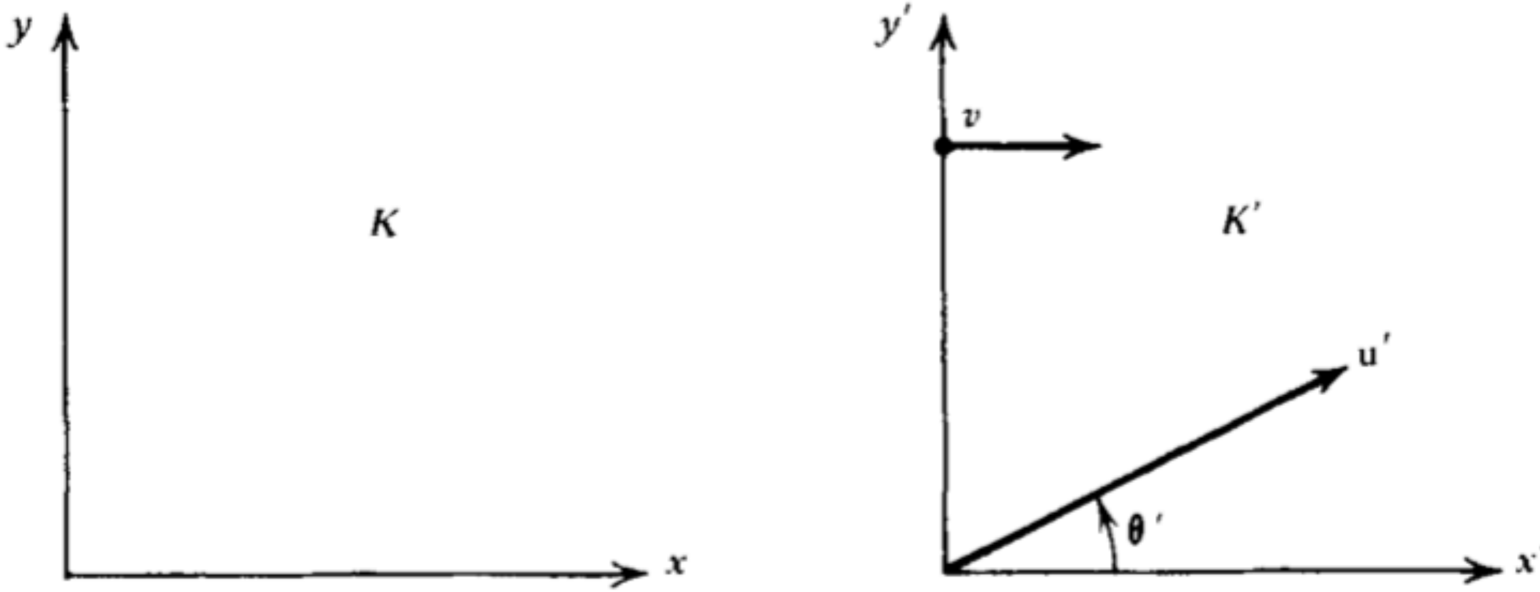
Transformations of velocities



$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma(dt' + v\,dx'/c^2)} = \frac{u'_x + v}{1 + vu'_x/c^2}$$

$$u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} \qquad u_z = \frac{u'_z}{\gamma(1 + vu'_x/c^2)}$$

Transformations of velocities



$$u_{\parallel} = \frac{u'_{\parallel} + v}{\left(1 + v u'_{\parallel}/c^2\right)}, \qquad u_{\perp} = \frac{u'_{\perp}}{\gamma\left(1 + v u'_{\parallel}/c^2\right)}$$

Components of **u** parallel and perpendicular to **v**

$$\tan\theta = \frac{u_{\perp}}{u_{\parallel}} = \frac{u' \sin\theta'}{\gamma(u'\cos\theta' + v)},$$

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + v/c)}$$

$$\cos\theta = \frac{\cos\theta' + v/c}{1 + (v/c)\cos\theta'}$$

Transformations of velocities
Beaming effect

For θ' = π/2, considering a photon emitted at right angles to v in K' →
$$\tan\theta = \frac{c}{\gamma v}$$
$$\sin\theta = \frac{1}{\gamma}.$$

For highly relativistic speeds γ >>1 →
$$\theta \sim \frac{1}{\gamma}$$

Consider photons are emitted isotropically in K'.
Half will have θ' >π/2 and other half will have θ' <π/2

In frame K the photons are concentrated in forward direction in a cone of 1/γ.
This is called **beaming effect**.

Transformations of velocities
Beaming effect

For θ' = π/2, considering a photon emitted at right angles to v in K' →

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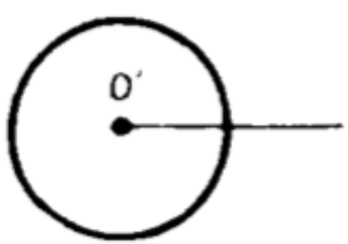
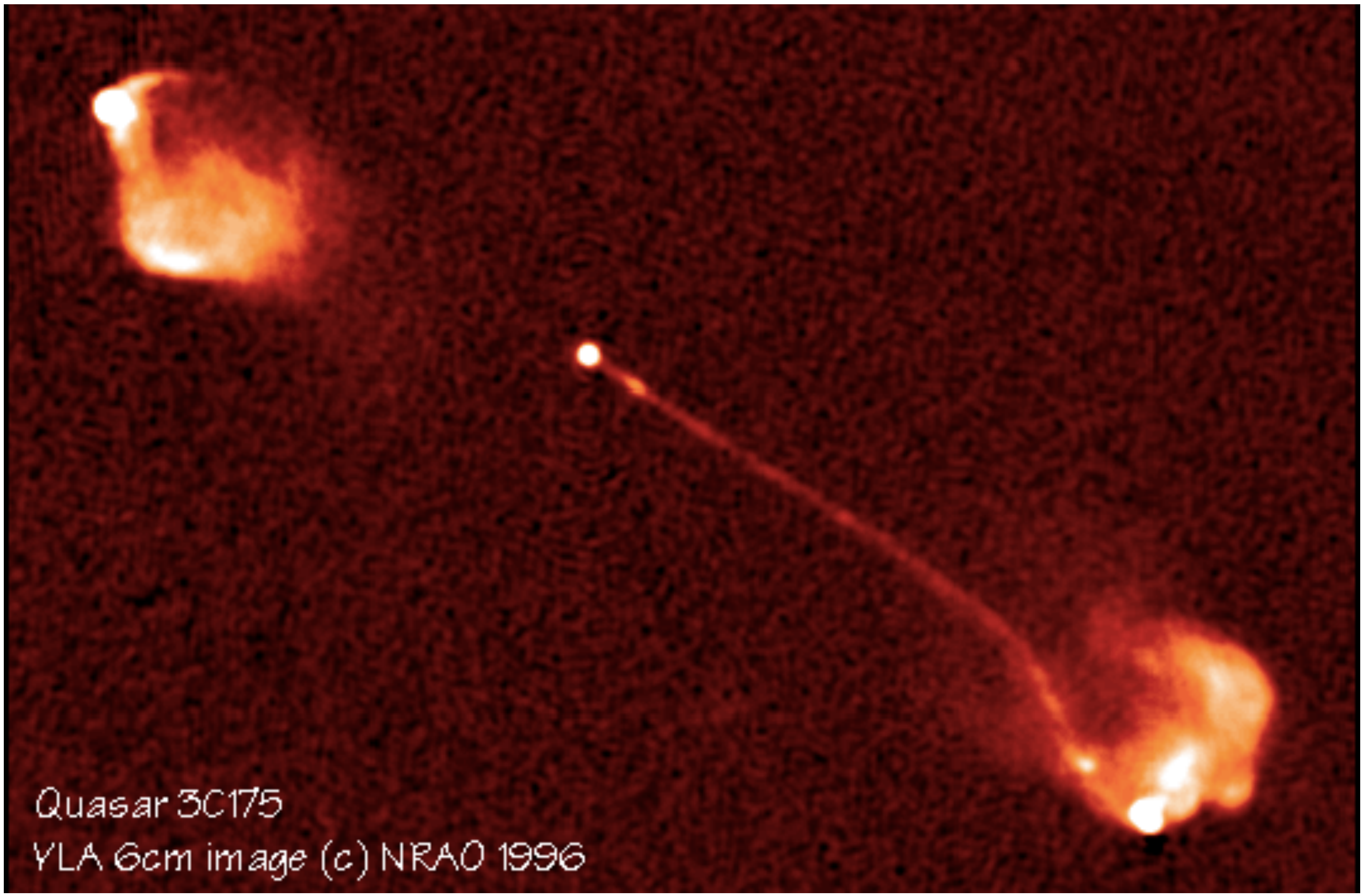


Isotropic emission: Rest frame K'

Beamed emission :K


Quasar 3C175
VLA 6cm image (c) NRAO 1996

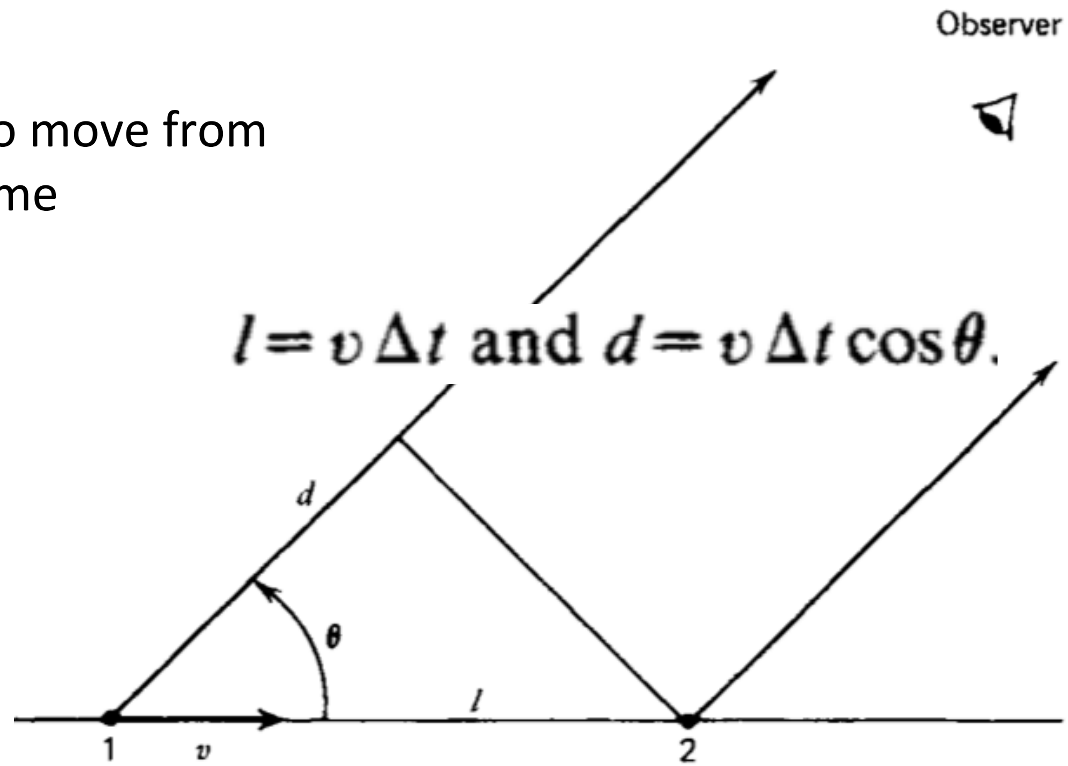
Doppler effect

Consider in rest frame of K
a source emits one period of radiation as it moves from point 1 to point 2

Time dilation implies, time taken to move from point 1 to point 2 in observer's frame

$$\Delta t = \frac{2\pi\gamma}{\omega'}$$

$$l = v\,\Delta t \text{ and } d = v\,\Delta t \cos\theta.$$



Difference in arrival time of the radiation emitted at 1 and 2

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t\left(1 - \frac{v}{c}\cos\theta\right)$$

Observed frequency

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma\left(1 - \frac{v}{c}\cos\theta\right)}$$

→ **Relativistic Doppler effect**

Proper time

Space and time have different values in different frames are separately subject to Lorentz transformation

Some quantities that are same in all Lorentz frames called Lorentz invariants

$$c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

Proper time $d\tau$ is unchanged under Lorentz transformation

Four vectors

One can find Lorentz transformation properties of other quantities as well. However four vectors have transformation properties identical to co-ordinates of events. So the treatment is less complicated.

$$c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

↓

quantities x,y,z,t an be formed in to a vector in four-dimensional space

Define

$$x^0 = ct$$
$$x^1 = x$$
$$x^2 = y$$
$$x^3 = z.$$

Space-time is a four-vector: $x^\mu = [ct, x]$
For μ=0,1,2,3

Four vectors

Four vectors −Four components that transform in a specific way under Lorentz transformation

Length of Four vectors is invariant i.e. same in every inertial system

Electromagnetism predicts that waves travel at c in vacuum.

Laws of electro magnetism must be Lorentz invariant.

Special relativity in one slide

Space-time is a four-vector: $x^{\mu} = [ct, \mathbf{x}]$

Four-vectors have Lorentz transformations between two frames with uniform relative velocity v:

$$x' = \gamma(x - \beta ct); \qquad ct' = \gamma(ct - \beta x)$$

$$\beta = v/c \text{ and } \gamma = 1/\sqrt{1-\beta^2}$$

Lengths of four vectors are Lorentz invariant

$$x^{\mu}x^{\nu} = c^2t^2 - |x|^2 = c^2t'^2 - |x'|^2 = s^2$$

Charge and Current densities

Under a Lorentz transformation a static charge q at rest becomes a charge moving with velocity v. This is a current.

A static charge density ρ at one frame becomes a current density J in other

Note: Charge is conserved by a Lorentz transformation

The charge/current four-vector is:

$$J^{\mu} = \rho dx^{\mu}/dt = [c\rho, \mathbf{J}]$$

The full Lorentz transformation is:

$$J'_x = \gamma(J_x - v\rho); \qquad \rho' = \gamma(\rho - v/c^2\, J_x)$$

Note: γ factor can be understood as a length contraction or time dilation affecting the charge and current densities

Electrostatic & vector potentials

A static charge density ρ is a source of an electrostatic potential V
A current density J is a source of a magnetic vector potential A

$$V = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho}{r} d\tau \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}}{r} d\tau$$

Under a Lorentz transformation a V becomes an A:

$$A'_x = \gamma(A_x - \frac{v}{c^2}V) \qquad V' = \gamma(V - vA_x)$$

The potential four-vector is

$$A^\mu = \left[\frac{V}{c}, \mathbf{A}\right]$$

Recap

Special theory of relativity

Length contraction (length of a moving rod appears smaller)

Time dilation (moving clock appears slower)

Transformation of velocities

Addition of velocities

Beaming effect

Energy of a moving body

Relativistic Doppler effect

Proper time

Four vectors

Recap

Special theory of relativity

Length contraction (length of a moving rod appears smaller) $L=\left(1-\frac{v^2}{c^2}\right)^{1/2}L_0$

Time dilation (moving clock appears slower) $T=t_2-t_1=\gamma(t_2'-t_1')=\gamma T_0$

Transformation of velocities $u_{\parallel}=\frac{u_{\parallel}'+v}{(1+vu_{\parallel}'/c^2)}, \quad u_{\perp}=\frac{u_{\perp}'}{\gamma(1+vu_{\parallel}'/c^2)}$

Addition of velocities

Beaming effect $\theta \sim \frac{1}{\gamma}$

Energy of a moving body $E_k=m_oc^2/\sqrt{(1+v^2/c^2)}$

Relativistic Doppler effect $\omega=\frac{2\pi}{\Delta t_A}=\frac{\omega'}{\gamma\left(1-\frac{v}{c}\cos\theta\right)}$

Proper time $c^2 d\tau^2=c^2 dt^2-(dx^2+dy^2+dz^2)$

Four vectors

Space-time is a four-vector: x^{μ} = [ct, x]
For μ=0,1,2,3

Minkowski

"…Space by it self and time by itself are doomed to fade away into mare shadows, and only a kind of union of the two will preserve an independent reality."

"Goings in the physical world are described by the geometrical structures in the space time"

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

⬇

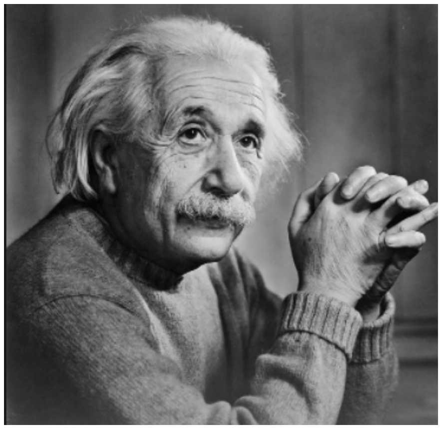
✓ Interval between two events is same in all inertial frames of reference.
It is invariant under Lorentz transformation.

✓ Lorentz transformation is nothing but rotation in space time.

"Minkowski took relativity out of special theory of relativity and presented us with an absolute picture of spatio-temporal activity" @ Penrose

"Minkowski's insights were the key to the discovery of General theory of relativity" @ Penrose

Special theory of relativity to general theory of relativity

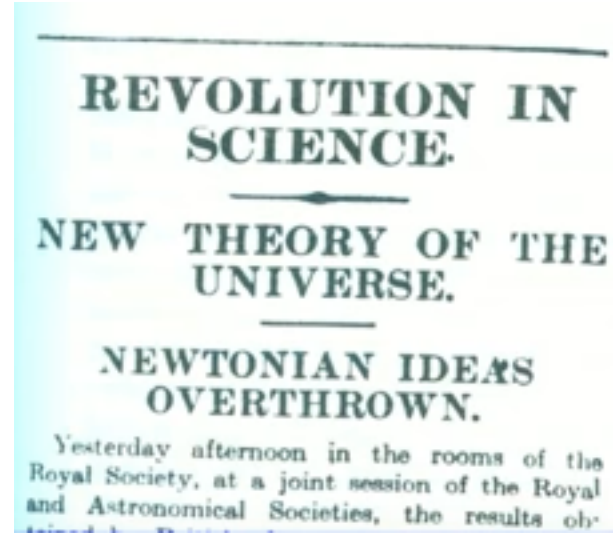


Relation between inertia and energy existed in special theory of relativity
But no relation between inertia and weight..
You can not switch off gravity..

But locally a freely falling body will not experience its weight.

Einstein's thought experiment
"No experiment can distinguish between uniform acceleration due to an engine and uniform to gravitational field."

Eddington



REVOLUTION IN SCIENCE.

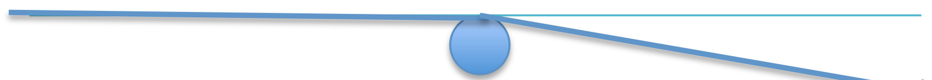
NEW THEORY OF THE UNIVERSE.

NEWTONIAN IDEAS OVERTHROWN.

Yesterday afternoon in the rooms of the Royal Society, at a joint session of the Royal and Astronomical Societies, the results ob-

1919 in a session of the Royal Society of London
Eddington verified Einstein's prediction of diffraction of light by the sun.
Light is deflected according to Einstein's law of Gravitation

"Fusion of two disconnected subjects, metric and gravitation can be considered as the most beautiful achievements of the general theory of relativity." Pauli



1.7 arc sec

Light has weight and would therefore be deflected by gravity

No experiment on Earth will be able to measure this deflection as it is too small and at that time technology would not have allowed it to be measured.

If you have a ray of light grazing the surface of the sun, then it would be deflected by 1.7 arc sec

Gravity modified Minkowski space time..

"Fusion of two disconnected subjects, metric and gravitation can be considered as the most beautiful achievements of the general theory of relativity." Pauli

Four vectors

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Electromagnetism predicts that waves travel at c in vacuum.

Laws of electro magnetism must be Lorentz invariant.

Special relativity in one slide
(Repeat)

Four vector- 1

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$$\beta = v/c \text{ and } \gamma = 1/\sqrt{1 - \beta^2}$$

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Charge and Current densities
(Repeat)

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Note: Charge is conserved by a Lorentz transformation

Four-vector 2
The charge/current four-vector is:

$$J^{\mu} = [c\rho, \mathbf{J}]$$

The full Lorentz transformation is:

$$J'_x = \gamma(J_x - v\rho); \qquad \rho' = \gamma(\rho - v/c^2\, J_x)$$

Note: γ factor can be understood as a length contraction or time dilation affecting the charge and current densities

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Four-vector 3

The potential four-vector is

$$A^\mu = \left[\frac{V}{c}, \mathbf{A}\right]$$

Under a Lorentz transformation a V becomes an A:

$$A'_x = \gamma(A_x - \frac{v}{c^2}V) \qquad V' = \gamma(V - vA_x)$$

Covarience of electromagnetic phenomenon

Continuity equation:

$$\nabla.\mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \partial_\mu J^\mu = 0$$

This shows that charge conservation is Lorentz invariant!

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Lorentz gauge condition

$$\frac{1}{c}\frac{\partial V}{\partial t} + \nabla.\mathbf{A} = 0 \qquad \partial_\mu A^\mu = 0$$

Covarience of electromagnetic phenomenon

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$$\frac{1}{c}\frac{\partial V}{\partial t} + \nabla.\mathbf{A} = 0 \qquad \partial_\mu A^\mu = 0$$

Poisson's equations

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \qquad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\partial_\mu^2 A^\mu = -\mu_0 J^\mu$$

Electric and Magnetic fields

The Lorentz force on a moving charge is,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

A static point charge is a source of an E field
A moving charge is a current source of a B field

Whether a field is E or B depends on the observer's frame

Lorentz transformation of E and B

Electric and magnetic field in terms of potentials can be written as

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla V \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz transformation of potentials

$$V' = \gamma(V - vA_x) \qquad A'_x = \gamma(A_x - \frac{v}{c^2}V)$$

Using this transformation and the Lorentz gauge condition the transformations of the electric and magnetic fields are (no derivation)

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + \frac{v}{c^2}E_z) \qquad B'_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

Lorentz transformation of E and B

A charge at rest has B = 0 and a spherically symmetric E field
A highly relativistic charge has $\beta \rightarrow 1$, $\gamma >> 1$

The electric field is

$$E'_x = E_x \ll |\mathbf{E}'| \qquad E'_y = \gamma E_y \qquad E'_z = \gamma E_z$$

The magnetic field is

$$B'_x = 0 \qquad B'_y = \gamma \frac{v}{c^2} E_z \qquad B'_z = -\gamma \frac{v}{c^2} E_y$$

Electromagnetic field Tensor

Electric field and magnetic field can be expressed as components of Electromagnetic field tensor in following form

$$F^{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

where $x = [ct, \mathbf{x}]$ $\quad A = [V/c, \mathbf{A}]$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

Maxwell's equation in terms of $F^{\mu\nu}$

Maxwell's equation with **source terms**

$$\frac{\partial F^{\mu\nu}}{\partial x_\nu} = J^\mu$$

M1 $\nabla.\mathbf{E} = \rho/\epsilon_0$ $\mu = 0,\ \nu = (1,2,3)$

M4 $\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \partial\mathbf{E}/\partial t)$ $\mu = 1,\ \nu = (2,3,0)$

Maxwell's equation without **source terms**

$$\frac{\partial F^{\mu\nu}}{\partial x_\sigma} + \frac{\partial F^{\sigma\mu}}{\partial x_\nu} + \frac{\partial F^{\nu\sigma}}{\partial x_\mu} = 0$$

M2 $\nabla.\mathbf{B} = 0$ $(\mu,\nu,\sigma) = (1,2,3)$

M3 $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ $(\mu,\nu,\sigma) = (0,1,2)(3,0,1)(2,3,0)$

Maxwell's equations are Lorentz invariant

Relativity and electromagnetic field

For the pure boost with velocity **v**=c**β**, equations can be written in the following form

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta}\times\mathbf{B}) \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \boldsymbol{\beta}\times\mathbf{E})$$

Pure Electric field is not Lorentz invariant

Pure Magnetic field is not Lorentz invariant

e.g. If the field is purely electric in one frame in another frame it will be a mixed electric and magnetic field

Any scalar formed with $F^{\mu\nu}$ represents function of E and B that is Lorentz invariant.

Relativity and electromagnetic field

Any scalar formed with $F^{\mu\nu}$ represents function of E and B that is Lorentz invariant.

$$F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2). \quad \Longrightarrow \quad \mathbf{B}^2 - \mathbf{E}^2 = \mathbf{B}'^2 - \mathbf{E}'^2$$

$$\det F = (\mathbf{E}\cdot\mathbf{B})^2. \quad \Longrightarrow \quad \mathbf{E}\cdot\mathbf{B} = \mathbf{E}'\cdot\mathbf{B}'$$

Fields of a uniformly moving charge

Fields of a charge moving with constant velocity v in the x-axis
In the rest frame of the charge the fields are

$$E'_x = \frac{qx'}{r'^3} \qquad B'_x = 0$$

$$E'_y = \frac{qy'}{r'^3} \qquad B'_y = 0$$

$$E'_z = \frac{qz'}{r'^3} \qquad B'_z = 0$$

$$r'^3 = (x'^2 + y'^2 + z'^2)^{3/2}.$$

Fields in the moving frame of charge

$$E_x = \frac{qx'}{r'^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y'}{r'^3} \qquad B_y = -\frac{q\gamma\beta z'}{r'^3}$$

$$E_z = \frac{q\gamma z'}{r'^3} \qquad B_z = \frac{q\gamma\beta y'}{r'^3}.$$

→

$$E_x = \frac{q\gamma(x - vt)}{r^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y}{r^3} \qquad B_y = -\frac{q\gamma\beta z}{r^3}$$

$$E_z = \frac{q\gamma z}{r^3} \qquad B_z = \frac{q\gamma\beta y}{r^3}$$

$$r^3 = \left[\gamma^2(x - vt)^2 + y^2 + z^2\right]^{3/2}.$$

Fields of a uniformly moving charge

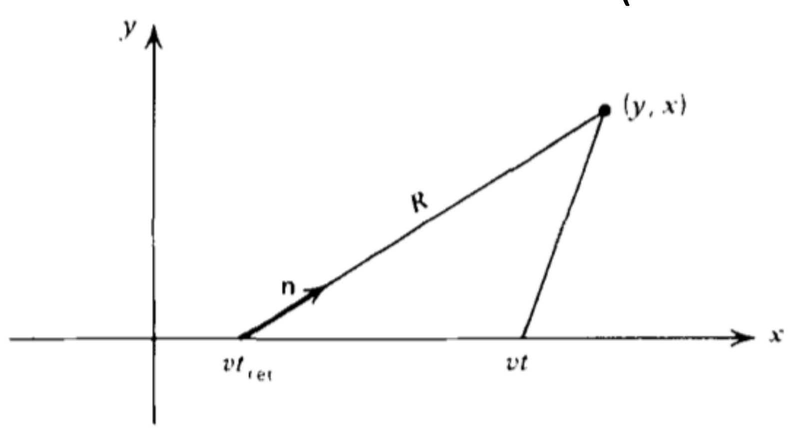
$$E_x = \frac{q\gamma(x - vt)}{r^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y}{r^3} \qquad B_y = -\frac{q\gamma\beta z}{r^3}$$

$$E_z = \frac{q\gamma z}{r^3} \qquad B_z = \frac{q\gamma\beta y}{r^3}$$

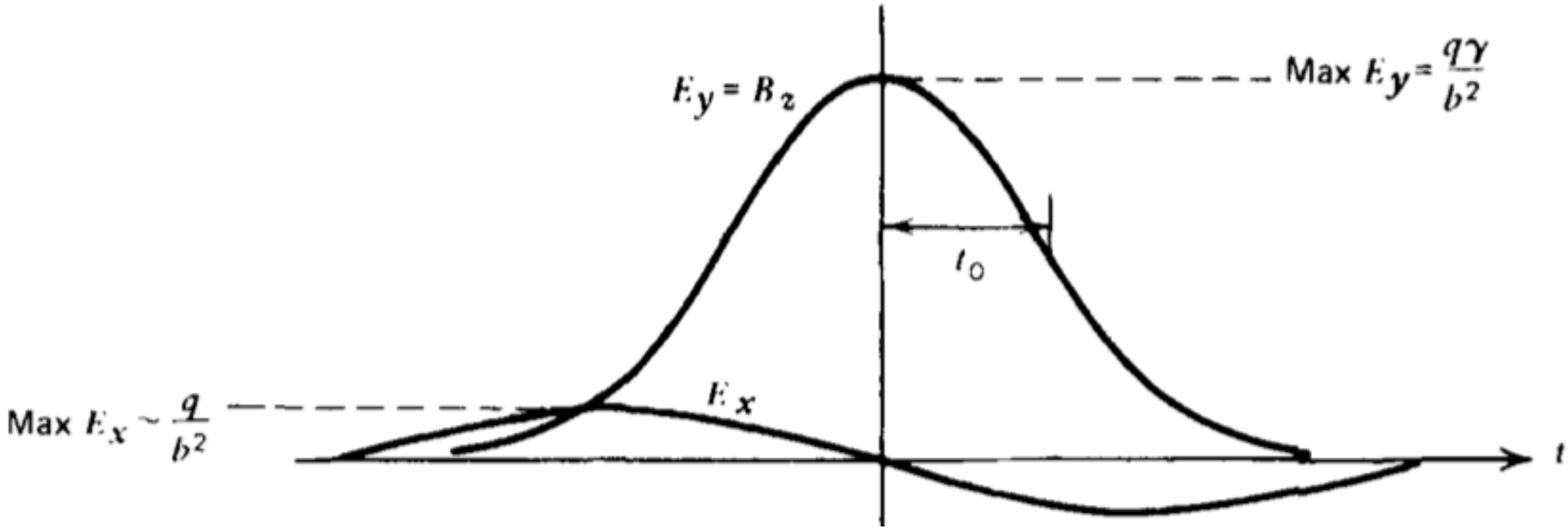
$$\mathbf{E} = q\left[\frac{(\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2)}{\kappa^3 R^2}\right]$$

E derived from Lienard-Wiechert potential in near field regime
(derived in 4.6 of R&L)



$$t_{ret} = t - \frac{R}{c}$$

Fields of a uniformly moving charge



Time dependence of fields E_x and E_y from a particle of uniform high velocity

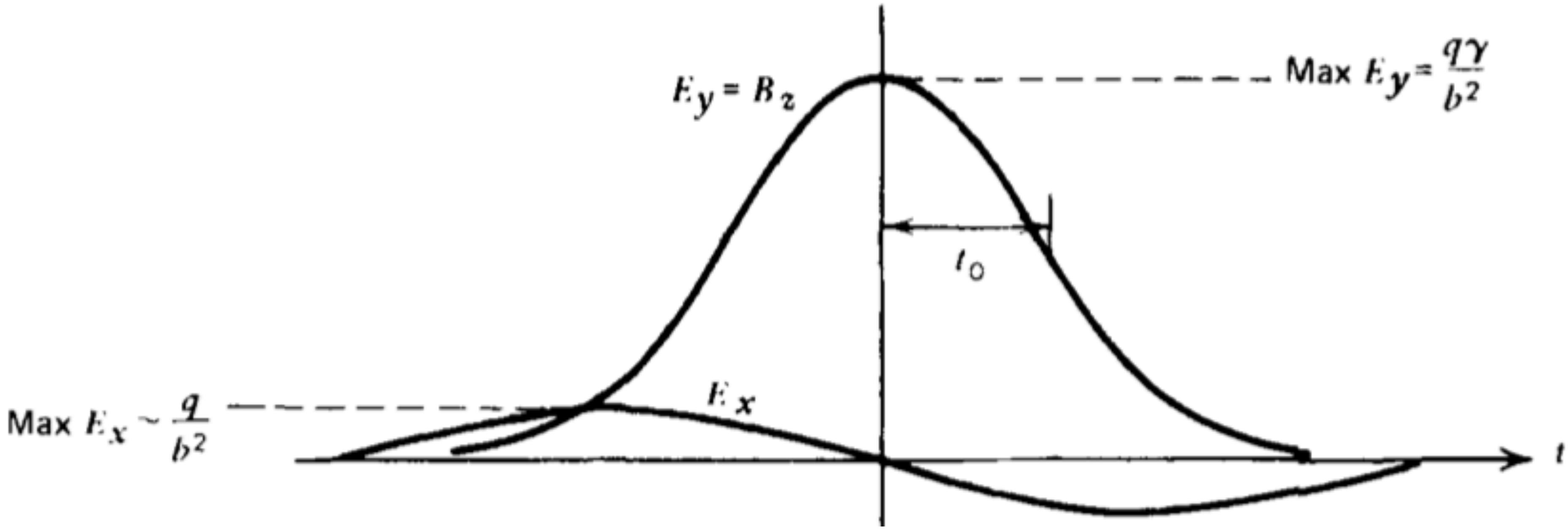
$$E_x = -\frac{qv\gamma t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_x = 0$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_y = 0$$

$$E_z = 0 \qquad B_z = \beta E_y$$

$$r^3 = \left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{3/2}.$$

Fields of a uniformly moving charge

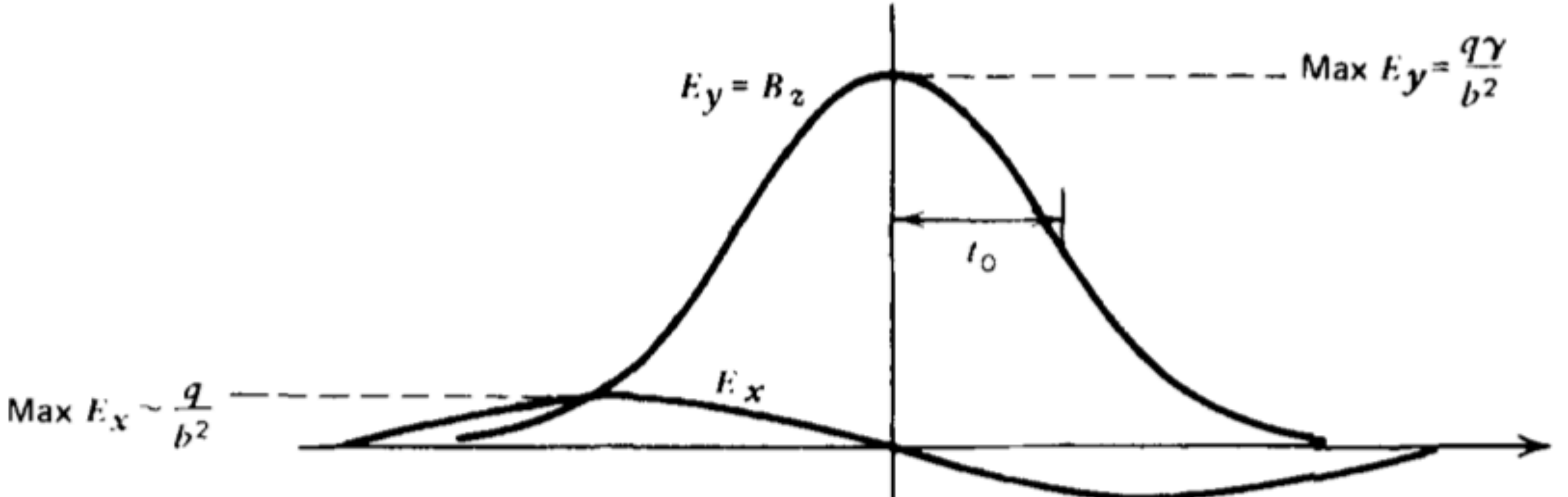


Time dependence of fields E_x and E_y from a particle of uniform high velocity

✓ Fields are strong when t~b/γv

✓ Fields of the moving charges are concentrated in the plane transverse to its motion into an angle of order of 1/γ

Fields of a uniformly moving charge



Time dependence of fields E_x and E_y from a particle of uniform high velocity

✓ Fields are mostly transverse (in y direction) since (Max E_x)/(Max E_y)= $1/\gamma$

✓ Field of a highly relativistic charge will appear as a pulse of radiation traveling in The same direction as the charge and confined to the transverse plane: Connection between fields of used in treatment of "method of virtual quanta" in relativistic bremssrahlung /synchrotron radiation.

Equivalent spectra

$$\hat{E}(\omega) = \frac{1}{2\pi}\int E_2(t)e^{i\omega t}\,dt$$
$$= \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}(\gamma^2 v^2 t^2 + b^2)^{-3/2} e^{i\omega t}\,dt$$

Integration in terms of modified Bessel function,

$$\hat{E}(\omega) = \frac{q}{\pi b v}\frac{b\omega}{\gamma v}K_1\left(\frac{b\omega}{\gamma v}\right)$$

Thus the spectrum is,

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2 = \frac{q^2 c}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^2\left(\frac{b\omega}{\gamma v}\right)$$

Equivalent spectra

The spectrum of virtual pulse,

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2 = \frac{q^2 c}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^2\left(\frac{b\omega}{\gamma v}\right)$$

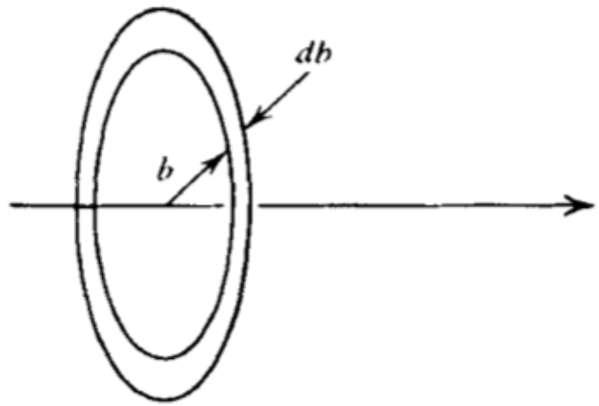
The spectrum starts to cut off for $\omega > \gamma v/b$

(From uncertainty principle since pulse is confined to time interval of ~$b/\gamma v$)

Equivalent spectra

The spectrum of virtual pulse,

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2 = \frac{q^2 c}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^2\left(\frac{b\omega}{\gamma v}\right)$$



Area element perpendicular to the velocity of a moving particle $dA = 2\pi b\,db$

Energy per unit frequency range

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{dW}{dA\,d\omega}\, b\,db$$

Equivalent spectra

Energy per unit frequency range

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{dW}{dA\, d\omega}\, b\, db$$

Lower limit to satisfy the description of field in classical electrodynamics and considering point charge,

e.g. b_{min}=radius of ion

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2} \int_x^{\infty} y K_1^2(y)\, dy$$

Considering

$$y \equiv \frac{\omega b}{\gamma v} \qquad x \equiv \frac{\omega b_{\min}}{\gamma v}$$

Integrating in terms of Bessel functions

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2} \left[x K_0(x) K_1(x) - \frac{1}{2} x^2 \left(K_1^2(x) - K_0^2(x) \right) \right]$$

Equivalent spectra

Energy per unit frequency range

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2}\left[xK_0(x)K_1(x) - \tfrac{1}{2}x^2\left(K_1^2(x) - K_0^2(x)\right)\right]$$

Two limits when ω is small and large

$$\frac{dW}{d\omega} = \frac{q^2c}{2v^2}\exp\left(-\frac{2\omega b_{\min}}{\gamma v}\right), \quad \omega \gg \frac{\gamma v}{b_{\min}}$$

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2}\ln\left(\frac{0.68\gamma v}{\omega b_{\min}}\right), \qquad \omega \ll \frac{\gamma v}{b_{\min}}$$

Emission from relativistic particles

The energy in as frame K moving with velocity −v with respect to the particle is

$$dW = \gamma\, dW' , \qquad dt = \gamma\, dt'$$

Total power emitted in frames K and K'

$$P = \frac{dW}{dt}, \qquad P' = \frac{dW'}{dt'} \quad \Rightarrow \quad P = P'.$$

Emitted power is Lorentz invariant for any emitter that emits with Front-back symmetry in its instantaneous rest frame

Emission from relativistic particles

$$P' = \frac{2q^2}{3c^3}|\mathbf{a}'|^2 \qquad\qquad P = \frac{2q^2}{3c^3}\vec{a}\cdot\vec{a}.$$

The power to be calculated in any frame by calculating **a** in that particular frame and squaring

$$a'_{\parallel} = \gamma^3 a_{\parallel},$$
$$a'_{\perp} = \gamma^2 a_{\perp}.$$

(Rybicki & Lightman 4.3)

Emission from relativistic particles

$$P' = \frac{2q^2}{3c^3}|\mathbf{a}'|^2$$

$$P = \frac{2q^2}{3c^3}\vec{a}\cdot\vec{a}.$$

The power and be calculated in any frame by calculating a in that particular frame and squaring

$$P = \frac{2q^2}{3c^3}\mathbf{a}'\cdot\mathbf{a}' = \frac{2q^2}{3c^3}\left(a_{\parallel}'^2 + a_{\perp}'^2\right)$$
$$= \frac{2q^2}{3c^3}\gamma^4\left(a_{\perp}^2 + \gamma^2 a_{\parallel}^2\right)$$

$$a_{\parallel}' = \gamma^3 a_{\parallel},$$
$$a_{\perp}' = \gamma^2 a_{\perp}.$$

End of Lecture 6

"It's not that I'm so smart, it's just that I stay with problems longer."
@ Einstein

Next lecture : 29th August

Topic of next Lecture:
Bremsstrahlung
(Chapter 5 of Rybicki & Lightman)