Electrodynamics and Radiative Processes I

Lecture 11 – Synchrotron Radiation

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IUCAA-NCRA Graduate School

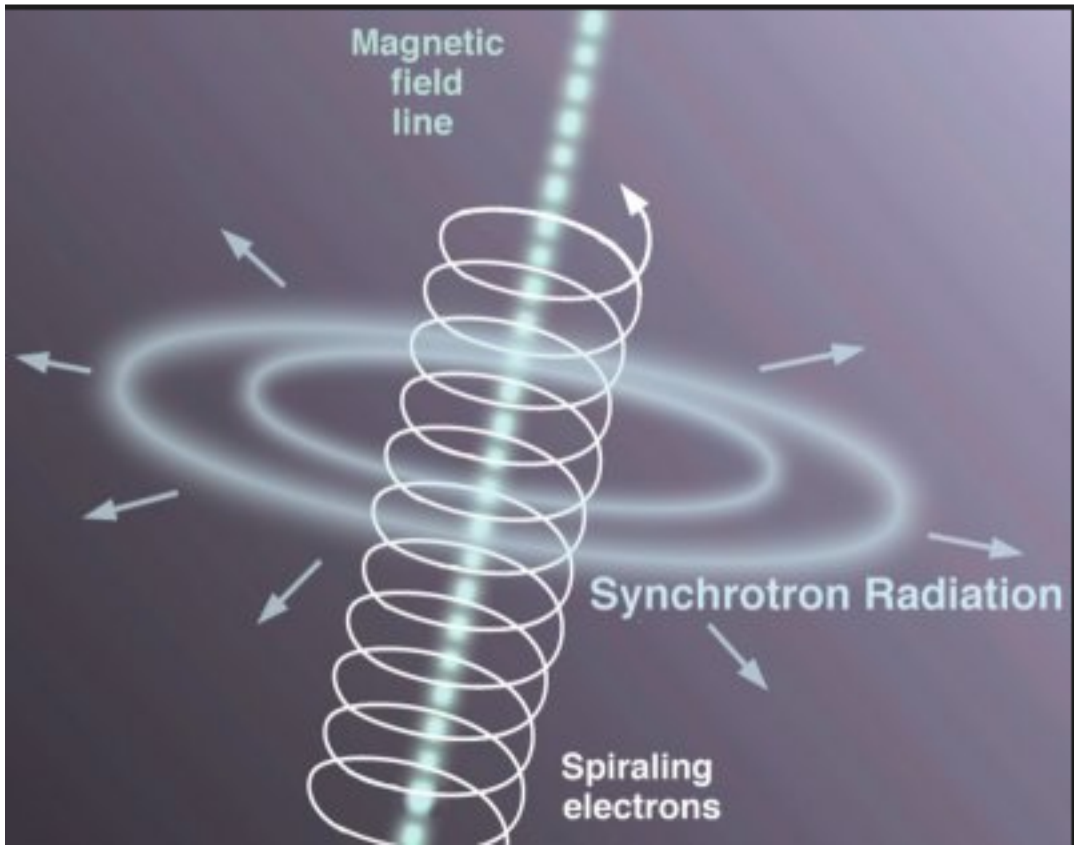
Reference :

1) Rybicki and Lightman
2) Ghisellini: http://www.brera.inaf.it/utenti/gabriele/total.pdf

Date : 14th[th] September 2018

Synchrotron Radiation(Recap)

Synchrotron Radiation is radiation from a charge moving relativistically that is accelerated by a magnetic field.



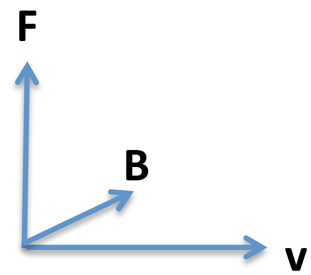
To understand synchrotron radiation let's first begin with the non-relativistic motion of a charge accelerated by a magnetic field : Cyclotron radiation

Cyclotron radiation

Let us take a charge (say q) and put it in uniform magnetic field B

Force F= ?

(If B is orthogonal to v)



Force F= Centripetal force

Larmor Radius /Gyro Radius

$r_L = ?$

Force $F = mv^2/r_L = m\,\omega_L r_L$

Cyclotron frequency

$\omega_{L} = ?$

Time period = ?

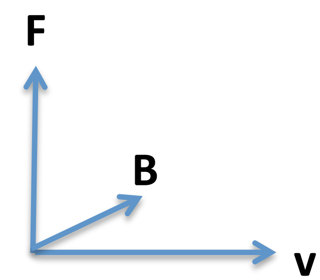
Cyclotron radiation summary

Let us take a charge (say q) and put it in uniform magnetic field B

Accelerated charged particle will radiate according to the Larmor formula $P = \frac{2q^2\dot{u}^2}{3c^3}$

Force $F = q\,\mathbf{v} \times \mathbf{B} = q\,v\,B$ (If B is orthogonal to v)

Force $F = q\,\mathbf{v} \times \mathbf{B} = q\,v\,B = mv^2/r_L$ = Centripetal force



Larmor Radius /Gyro Radius $\boxed{r_L = mv/qB}$

Force $F = mv^2/r_L = m\,\omega_L r_L$

Cyclotron frequency $\boxed{\omega_L = qB/m}$

$\nu_L = \omega_L/2\pi = qB/2\pi m = 2.8$ MHz per Gauss for electron

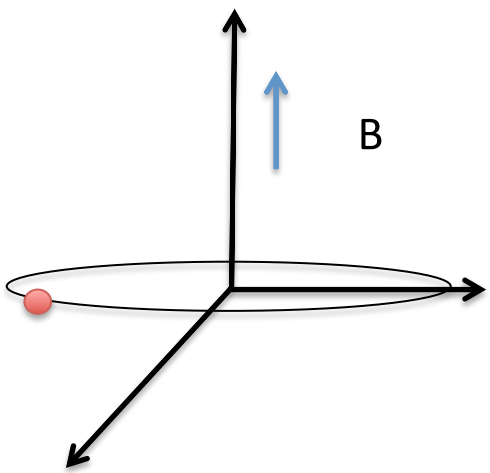
Frequency is independent of path radius and particle velocity

Time period $\boxed{T = 2\pi/\omega_L = 2\pi m/qB}$

Power spectra will peak at a single frequency

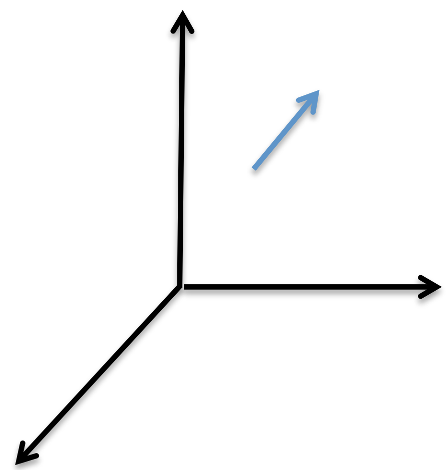
Cyclotron radiation
Polarization

B is perpendicular to LOS



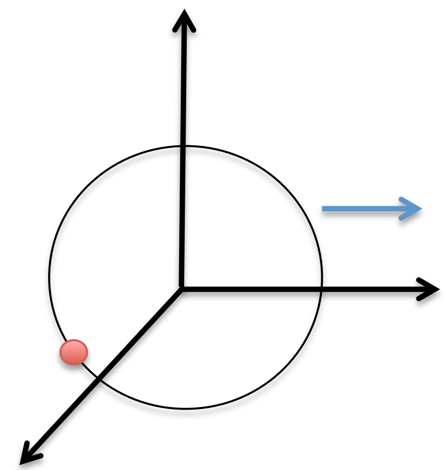
Linear Polarization

B is at an angle to LOS



Elliptical Polarization

B is parallel to LOS

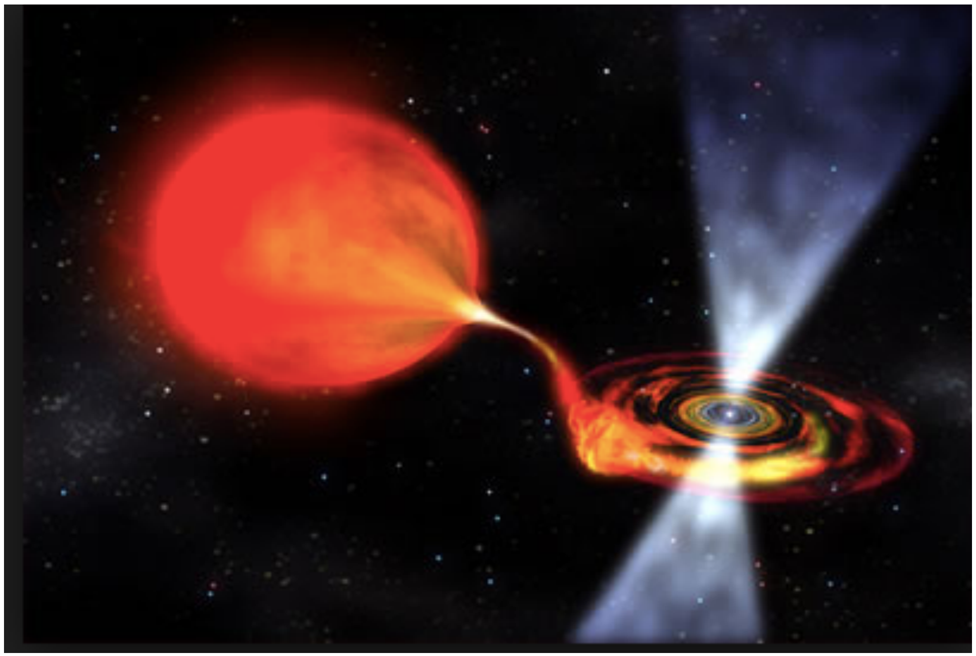


Circular Polarization

Polarization measurement to infer B strength and its orientation

Cyclotron radiation
Astrophysical application
Discovered ~ 40 years back

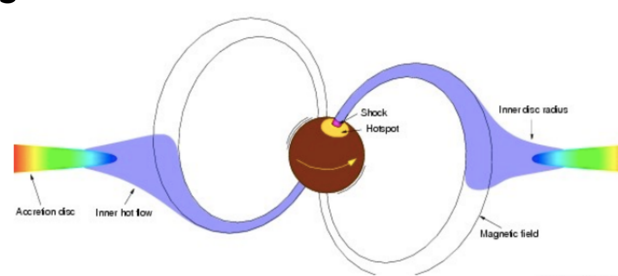
Cyclotron lines from the accreting x-ray pulsars



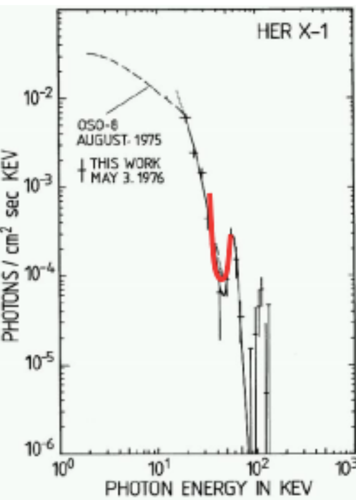
In 1977 J. Trumper identified a cyclotron emission line in the accreting pulsar Hercules X-1

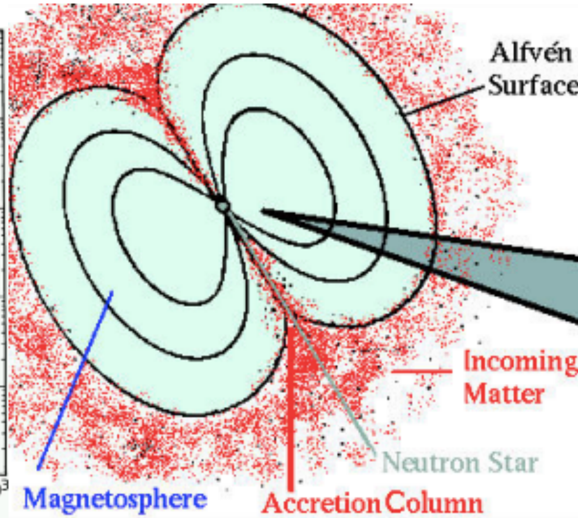
Trumper proposed : hot electrons around neutron star magnetic poles are rotating around a strong B field of ~5×10^{12} Gauss, giving rise to an absorption line at ~40 keV.

Directly probe the magnetic fields of the neutron stars
Probe geometry
Seen in more than 30 sources
Simulations + Observations

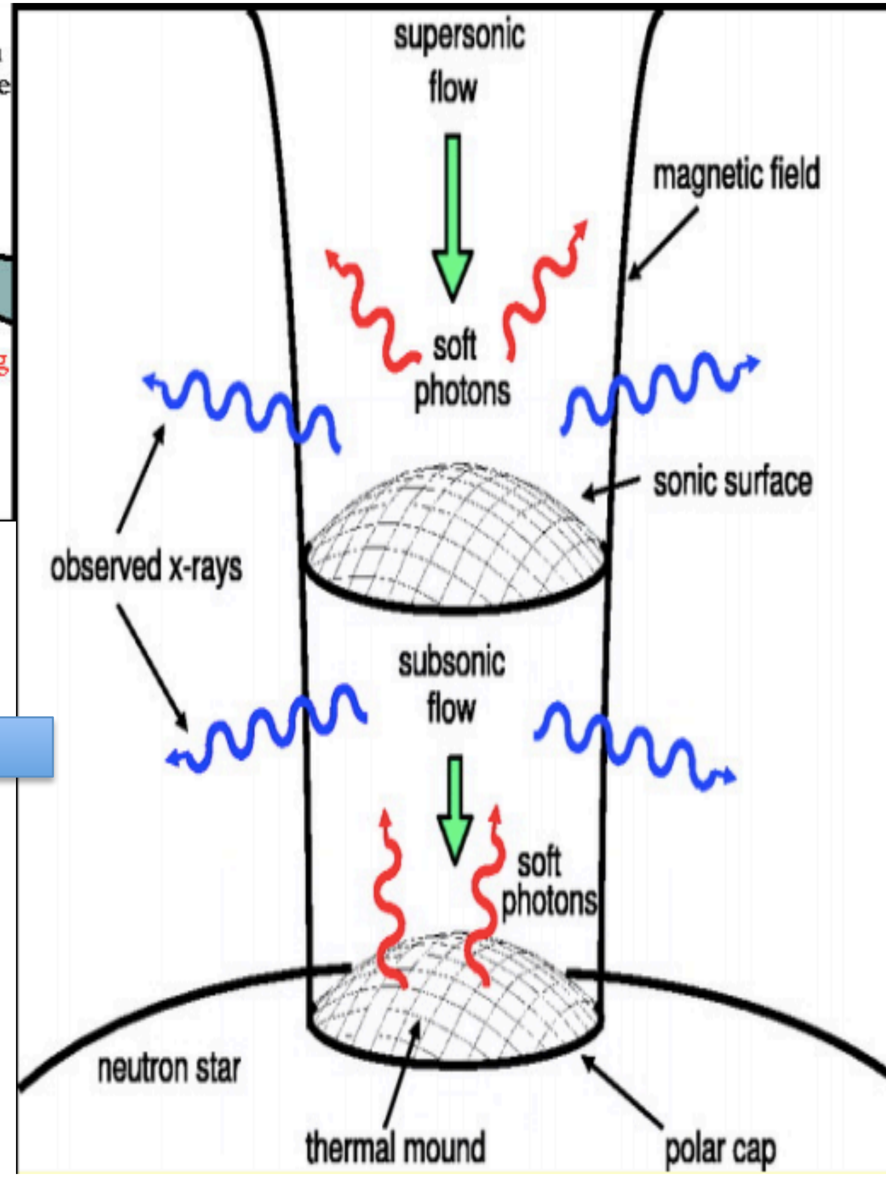


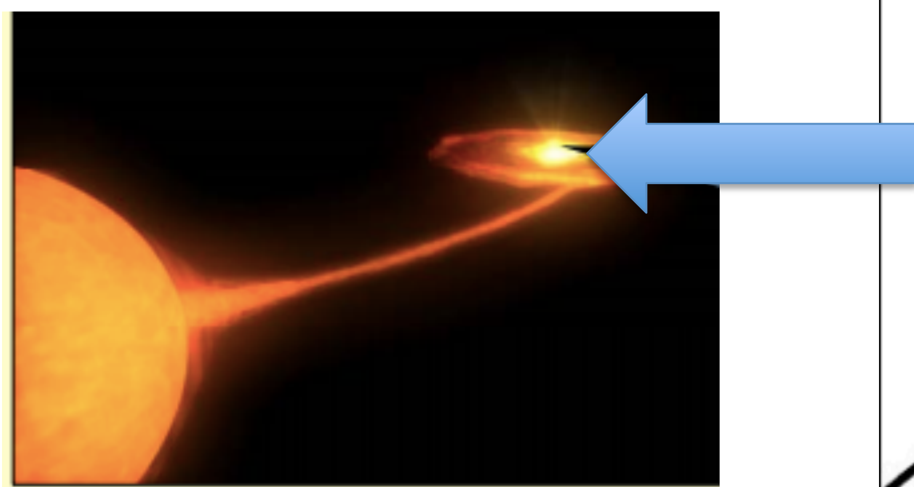
Observation





Modeling





Refer to :https://www.cosmos.esa.int/documents/13611/404108/200808_Schoenherr.pdf/ecff8c8e-f1e7-4f30-b3c8-66d682e20a13

Synchrotron Radiation
In Astrophysics

Magnetic fields and relativistic particles are prerequisite for synchrotron radiation in astrophysics.

So synchrotron emission is seen in a wide variety of environments.

Typical magnetic field strengths

Location	**Magnetic filed (Gauss)**
Interstellar medium	10^{-6}
Stellar atmosphere	1
Black hole	10^{4}
White dwarf	10^{2}
Neutron star	10^{12}
Earth	0.3

Relativistic effects: from Cyclotron to Synchrotron Radiation

Assumption $v<<c$ (non relativistic particles) **for Cyclotron**

Now we describe what happens to the radiation of a charge accelerated in a B field when the speeds approach c **for Synchrotron**

Review Relativistic effects discussed in Lecture 5

Lorentz transformations of time: $\Delta t = \Delta t' \gamma$

Lorentz transformations of Frequency: $\nu = \nu'/\gamma$

Relativistic effects: from Cyclotron to Synchrotron Radiation

Cyclotron → **Synchrotron**

Larmor Frequency → Frequency of Gyration

$\nu_L = \omega_L / 2\pi = qB/2\pi m$ → $\nu_B = \omega_B / 2\pi = qB/2\pi m\gamma$

Larmor radius → Radius of Gyration

$$r_L = \frac{mv}{qB} \rightarrow r_L = \sqrt{\frac{2mV}{qB^2}}$$

→

$$r_L = \sqrt{\frac{(\gamma+1)mV}{qB^2}} = \frac{\gamma m v}{qB}$$

Period of rotation → Period of rotation

$T = 2\pi/\omega_L = 2\pi m/qB$ → $T = 2\pi/\omega_B = 2\pi m\gamma/qB$

The period depend on particle velocity (Lorentz factor gamma) and as the velocity approaches c, the period increases.

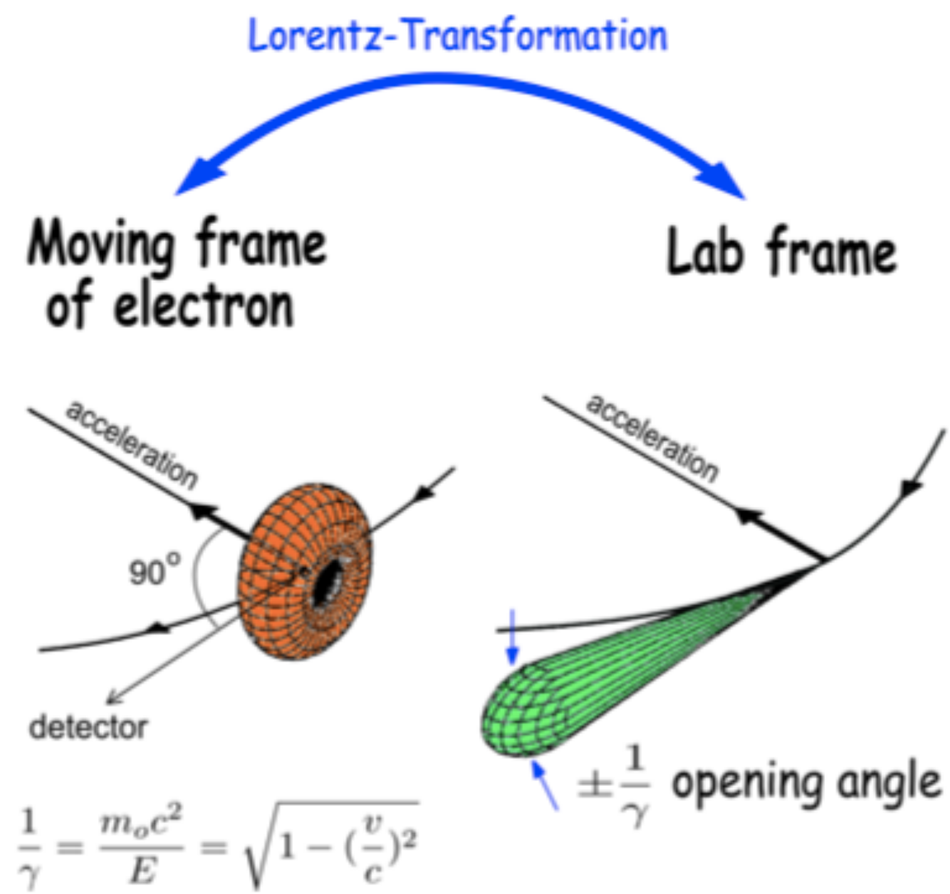
Synchrotron Radiation
Emission pattern

A relativistic electron moving around a B field.

Cyclotron to Synchrotron:

- start with the radiation pattern in the electron rest frame (where we know the radiation pattern)

- then we do a Lorentz transformation from the rest frame to the lab frame.



Synchrotron Radiation

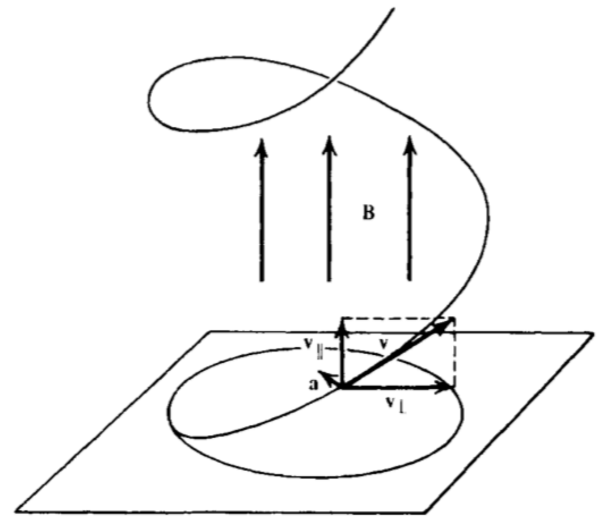
Synchrotron radiation: Motion of ultra-relativistic particles around the magnetic field lines

Consider a particle of mass m and charge q

Equations of Motion of a particle with relativistic velocity:

Change of relativistic momentum dp/dt

$$\frac{d}{dt}(\gamma m\mathbf{v}) = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$



γ is constant → |v| constant

$$m\gamma\frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

Force on the particle is perpendicular to the motion.

Synchrotron Radiation

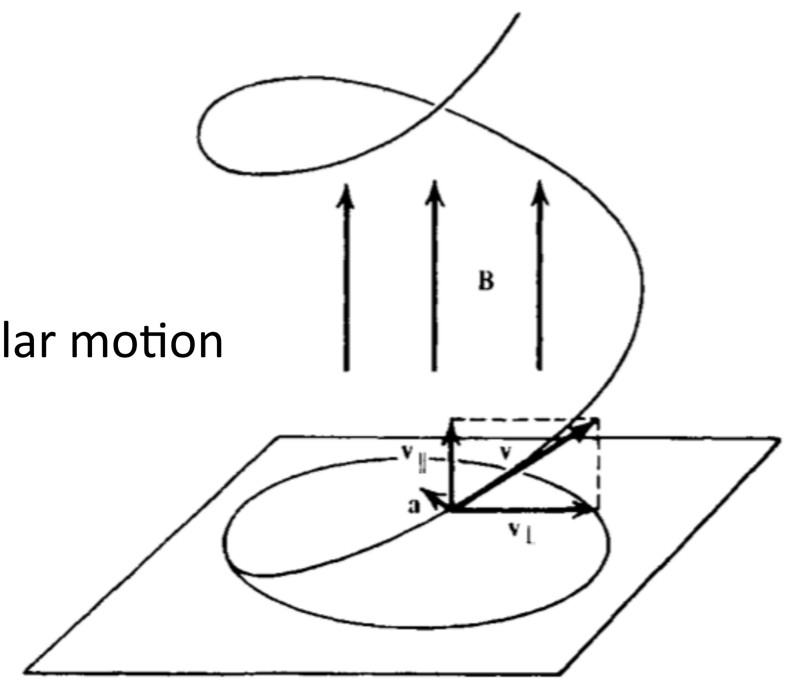
Helical Motion:

$$m\gamma \frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

$$\frac{d\mathbf{v}_{\parallel}}{dt} = 0, \qquad \frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{\gamma mc}\mathbf{v}_{\perp}\times\mathbf{B}$$

Separating the velocity components along the field and in a plane perpendicular to the field

Parallel component of v is constant
But |v|=constant
So perpendicular component of v is constant



Motion of the particle is combination of circular motion
And uniform motion along the field

Helical motion of the particle

Synchrotron Radiation

$$m\gamma \frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v} \times \mathbf{B}$$ =Force = mv^2/r

r is the radius of the orbit → radius of gyration

α is angle between field and velocity → pitch angle

$\pi/2$ for motion perpendicular to fields

$$\omega_B = \frac{qB}{\gamma mc}$$

↓

Calculate for different magnetic filed values (e.g. typical ISM, cosmic ray, neutron star etc)

Synchrotron Radiation

$$\omega_B = \frac{qB}{\gamma mc}$$

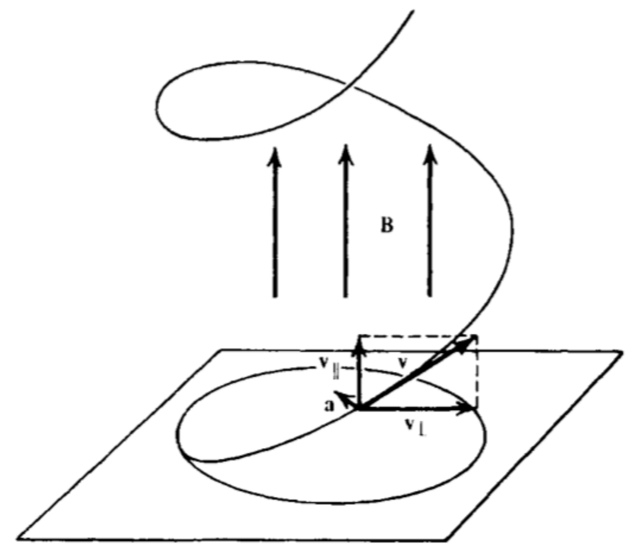
For ISM considering $B \sim 10^{-6}$ G and $\gamma=1$ $\omega_B \sim 30$ Hz

Knowing the $\omega_B < 1$ Hz for cosmic ray electrons → estimate the field strength

Synchrotron Radiation
(Total power radiated)

Total emitted radiation (From Lecture 7)

In the case of cyclotron, as well as (non-rel) Bremsstrahlung we saw that we can use the Larmor's formula (Lecture 3) to calculate the power emitted by an accelerated charge:



$$P = \frac{2q^2}{3c^3}\mathbf{a}'\cdot\mathbf{a}' = \frac{2q^2}{3c^3}\left(a_{\parallel}'^2 + a_{\perp}'^2\right)$$

$$= \frac{2q^2}{3c^3}\gamma^4\left(a_{\perp}^2 + \gamma^2 a_{\parallel}^2\right)$$

$a_{\perp} \rightarrow \omega_B v_{\perp}$; $a_{\parallel} \rightarrow$ zero

Lorentz transformation of the acceleration

$$a_{\parallel}' = \gamma^3 a_{\parallel},$$

$$a_{\perp}' = \gamma^2 a_{\perp}$$

Synchrotron Radiation
(Total power radiated)

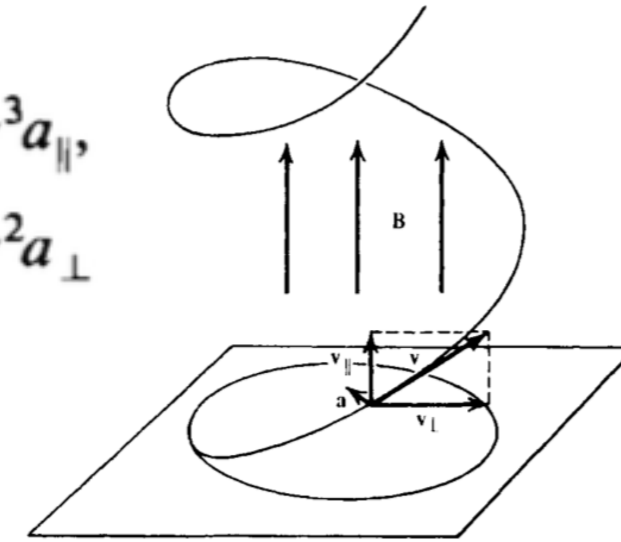
Total emitted radiation (From Lecture 7)

$$P = \frac{2q^2}{3c^3}\mathbf{a}'\cdot\mathbf{a}' = \frac{2q^2}{3c^3}\left(a_{\parallel}'^2 + a_{\perp}'^2\right)$$

$$= \frac{2q^2}{3c^3}\gamma^4\left(a_{\perp}^2 + \gamma^2 a_{\parallel}^2\right)$$

$a_{\perp} \rightarrow \omega_B v_{\perp}$, $a_{\parallel} \rightarrow$ zero

$$a'_{\parallel} = \gamma^3 a_{\parallel},$$
$$a'_{\perp} = \gamma^2 a_{\perp}$$



$$P = \frac{2q^2}{3c^3}\gamma^4\frac{q^2B^2}{\gamma^2m^2c^2}v_{\perp}^2 \longrightarrow P = \frac{2}{3}r_0^2c\beta_{\perp}^2\gamma^2B^2$$

Total emitted radiation from charged particles with velocity v

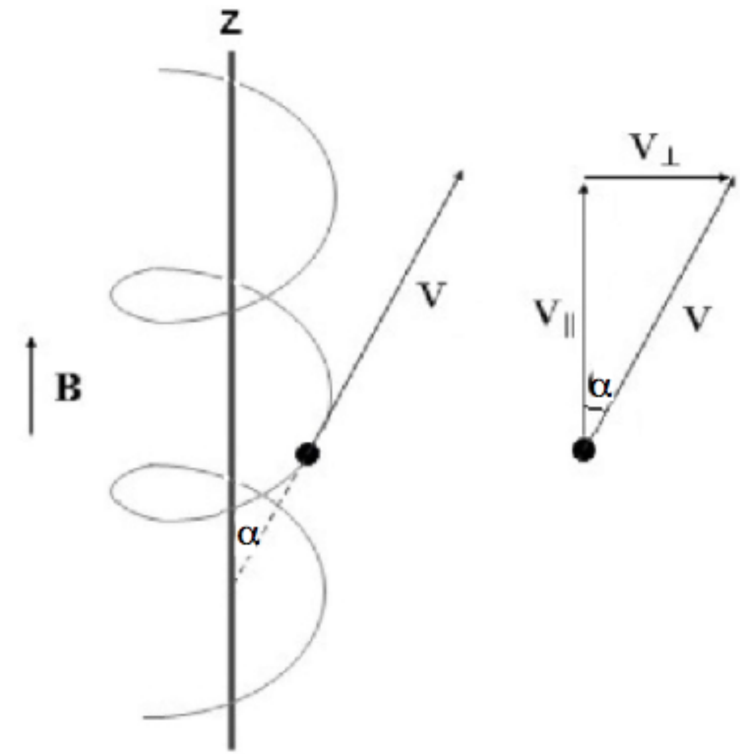
Synchrotron Radiation
(Total power radiated)

Total emitted radiation

$$P = \frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2$$

We have many particles each having a pitch angle. So the perpendicular velocity needs to be averaged over all pitch angles (α).

$$\langle \beta_\perp^2 \rangle = \frac{\beta^2}{4\pi} \int \sin^2\alpha \, d\Omega = \frac{2\beta^2}{3}$$



Synchrotron Radiation
(Total power radiated)

Total emitted radiation

$$P = \frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2$$

We have many particles each having a pitch angle. So the perpendicular velocity needs to be averaged over all pitch angles (α).

$$\langle \beta_\perp^2 \rangle = \frac{\beta^2}{4\pi} \int \sin^2\alpha \, d\Omega = \frac{2\beta^2}{3}$$

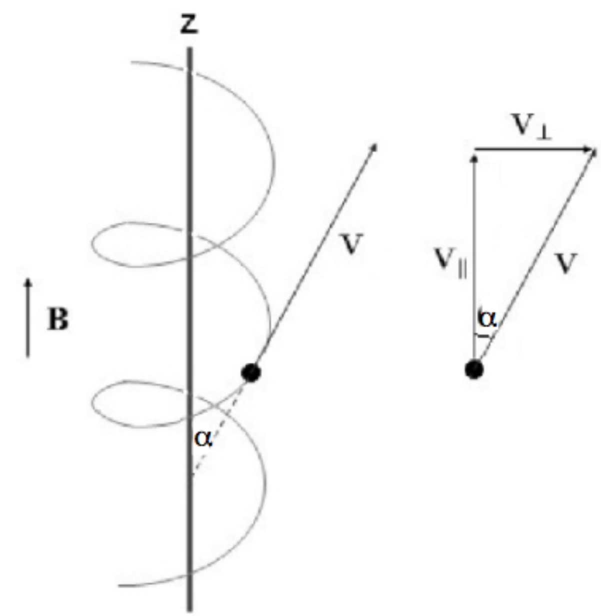
Total emitted radiation

$$P = \left(\tfrac{2}{3}\right)^2 r_0^2 c \beta^2 \gamma^2 B^2,$$

$$\sigma_T = 8\pi r_0^2/3$$
$$U_B = B^2/8\pi$$

Total emitted radiation for electrons

$$P = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 U_B.$$



Synchrotron Radiation

(Total power radiated)

$$P = \frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2$$

→ Valid for electron only

$$P = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 U_B$$

→ Valid for electron only

The formula is valid only for electrons emitting synchrotron radiation. The reason why we write this formula only for electrons is because in basically all astrophysical cases you have electron synchrotron. This is because electrons become relativistic much more quickly than protons as they are easier to accelerate.

Synchrotron Radiation
(Total power radiated)

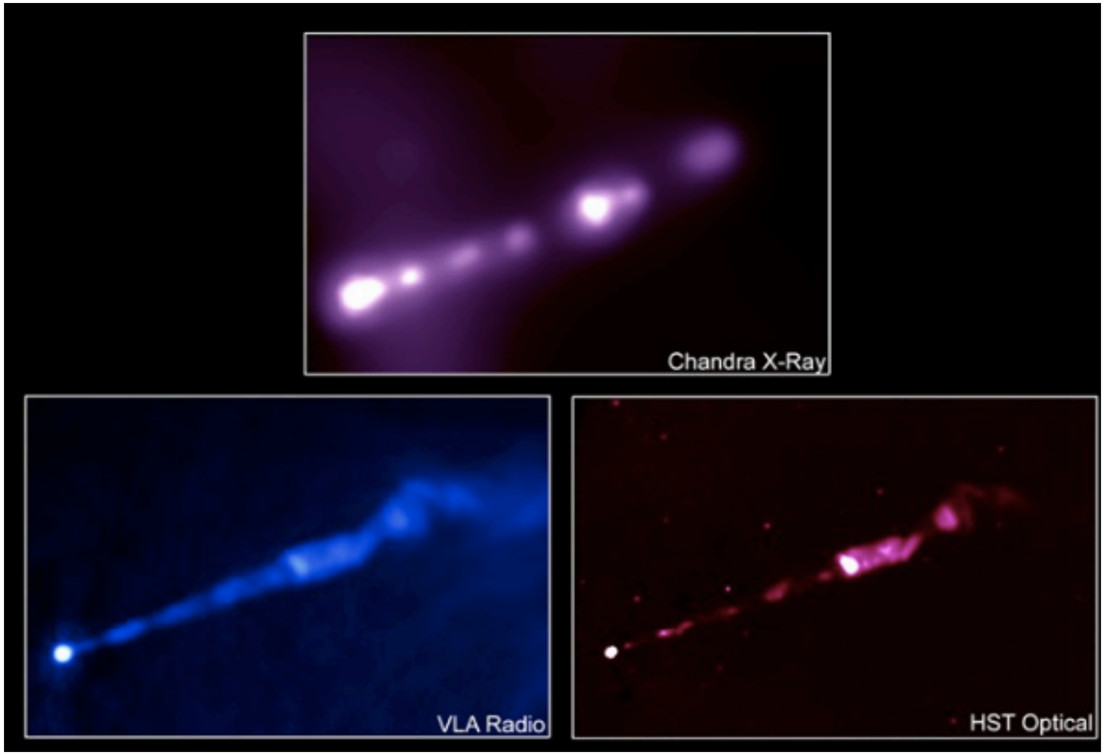
Suppose the protons of the LHC are accelerated up to an energy of 7 TeV and then they are left to cool down due to synchrotron emission. On which timescale do they cool down?

Time scale ~ (Proton energy)/ (Synchrotron power) ~ few days

Time scale ~ (Electron energy)/ (Synchrotron power) ~ nano seconds

Electrons cools down by a factor of $\sim 10^{13}$ times faster than protons

Synchrotron in Astrophysics

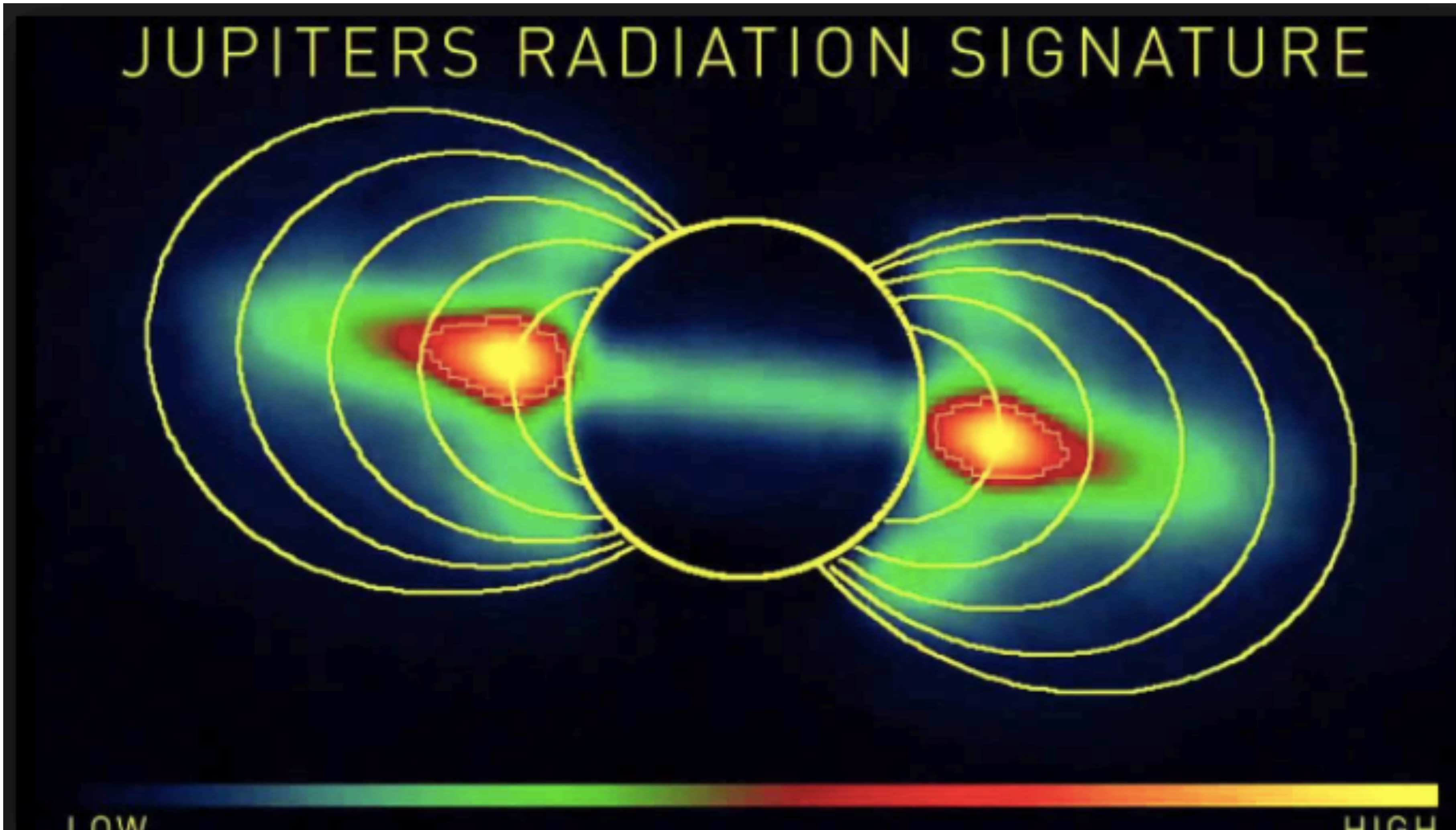


Astrophysical Jets

Courtesy :Alessandro Patruno

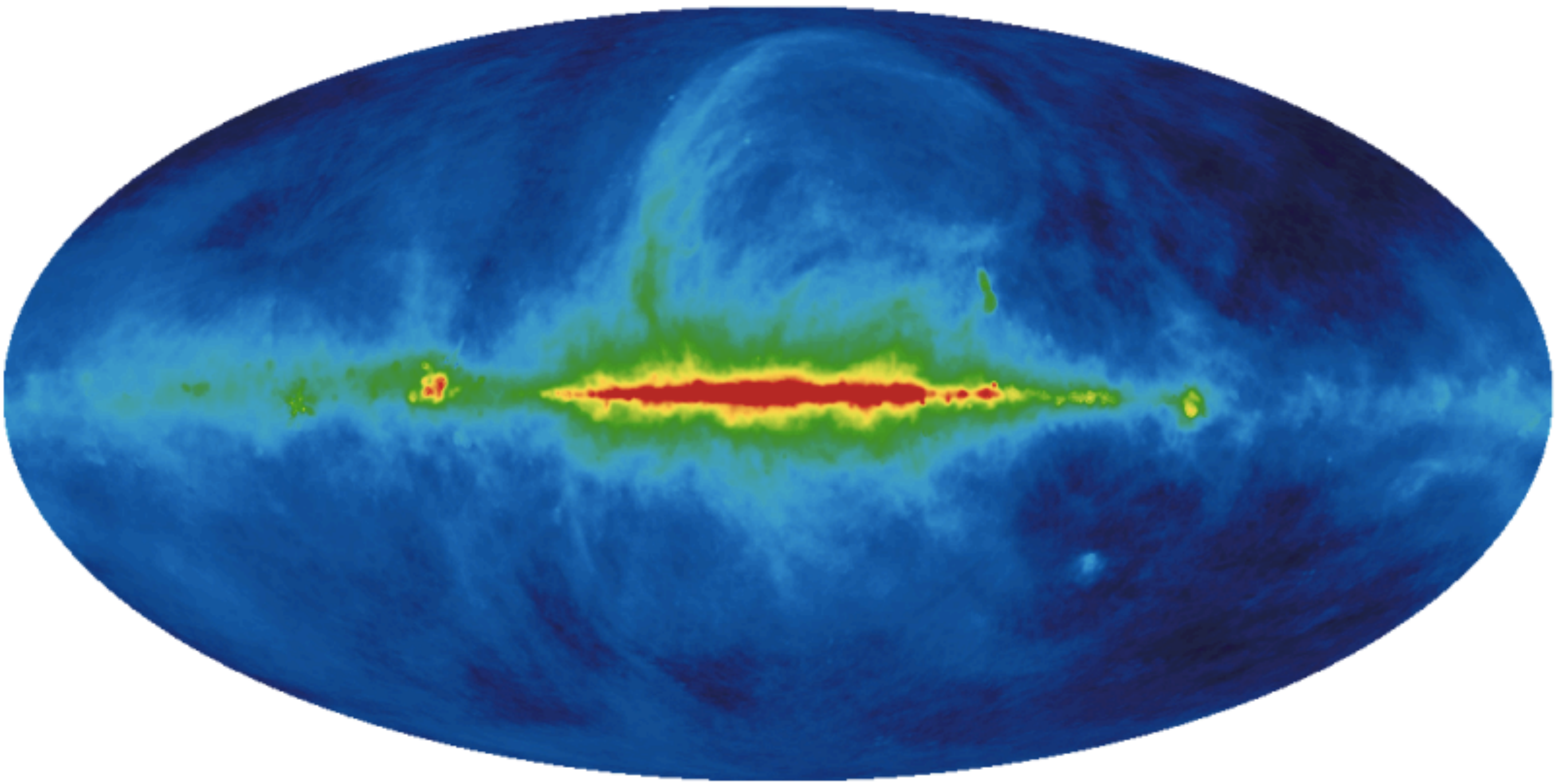
“Astrophysical jets are most likely generated by relativistic particles being launched close to a black hole (or even a neutron star when in a binary). Such particles are thought to be electron/positron pairs which then spiral along B field lines and generate synchrotron radiation. However, we also know that cosmic rays most likely come from Active Galactic Nuclei, where strong B fields around supermassive black holes launch streams of ultra-relativistic particles which include protons. So it’s still unclear whether jet emission is due to leptons or hadrons.”

Synchrotron Radiation
Jupiter's Belt

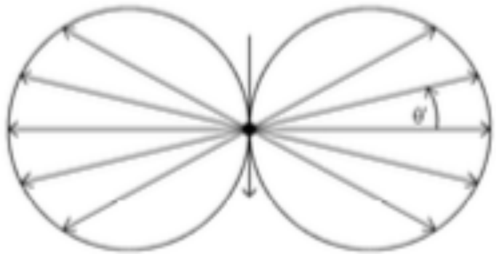


Galactic Synchrotron

Haslam et al. map at 408 MHz for Galactic synchrotron emission



Synchrotron Radiation
Emission pattern



Rest frame of electron

Laboratory frame of reference

Beaming :
Important to make a distinction between emitted radiation and received radiation.
Received radiation will be such that the observer can see it only when the narrow beam points towards the observer
→radiation appears to be concentrated on a narrow cone.
Observer will see radiation from a particle only for a small fraction $2/\gamma$ of its orbit.

Observer will see pulse of radiation confined to a time much smaller than its gyration period.

Spectrum will be spread over region much broader than $\omega_B/2\pi$

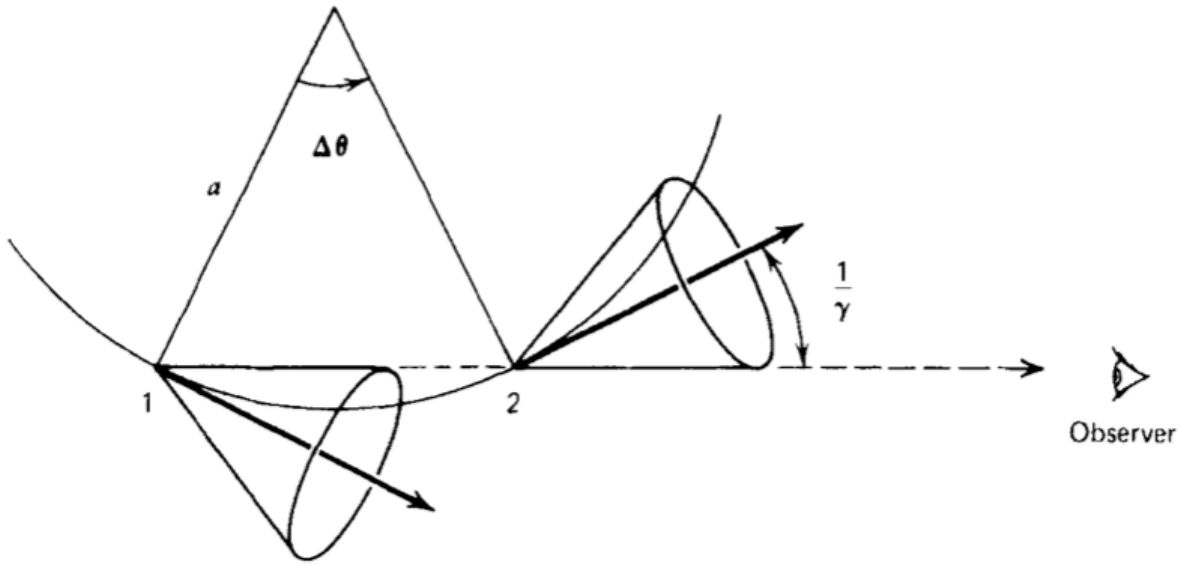
Synchrotron Radiation
(spectrum)

The spectrum of synchrotron radiation must be related to detailed variation of electric field seen by an observer

Because of beaming, emitted radiation appear to be concentrated about particle's velocity

Angular distribution of radiation emitted by a particle with perpendicular acceleration and velocity

Synchrotron Radiation
(spectrum)



Emission cones at various points of an accelerated particles trajectory

Observer will see pulse from point 1 and 2 along the particles path, where these points are such that the cone of emission of angular width $1/\gamma$ includes the direction of observation

$$a = \Delta s / \Delta\theta$$

$\Delta\theta = 2/\gamma$ (from geometry) $\Delta s = 2a/\gamma$

Synchrotron Radiation
(spectrum)

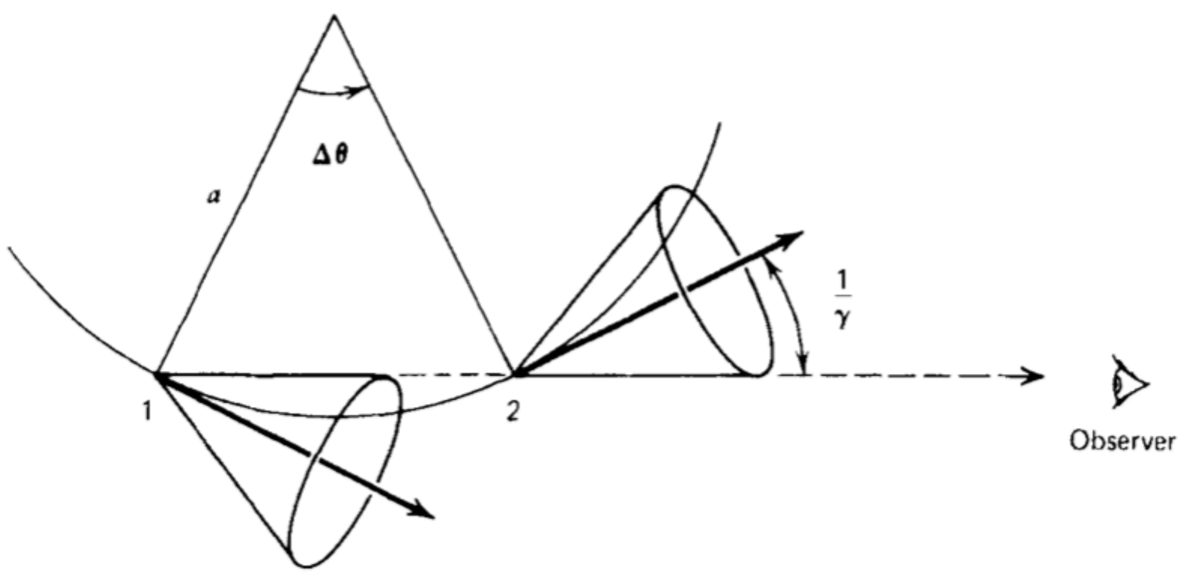
Equation of motion $$\gamma m \frac{\Delta \mathbf{v}}{\Delta t} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

Since |Δv|=v Δθ and Δs=vΔt we have $$\frac{\Delta\theta}{\Delta s} = \frac{qB\sin\alpha}{\gamma m c v} \longrightarrow a = \frac{v}{\omega_B \sin\alpha}$$

$$\omega_B = \frac{qB}{\gamma m c}$$

Δs=2a/γ

$$\Delta s \approx \frac{2v}{\gamma \omega_B \sin\alpha}$$

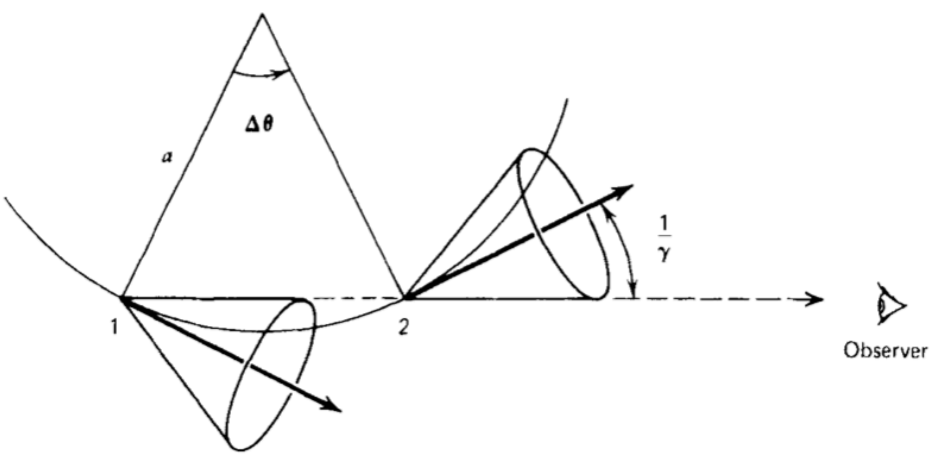


Synchrotron Radiation
(spectrum)

Times t_1 and t_2 at which particle passes points 1 and 2 are such that $\Delta s = v(t_2 - t_1)$

$$t_2 - t_1 \approx \frac{2}{\gamma \omega_B \sin\alpha}$$

Times t_1^A and t_2^A be the arrival times of radiation at the point of observation, $t_1^A - t_2^A$ is less than $t_1 - t_2$ by $\Delta s/c$ (time for the radiation to move Δs)



$$\Delta t^A = t_2^A - t_1^A = \frac{2}{\gamma \omega_B \sin\alpha}\left(1 - \frac{v}{c}\right)$$

Synchrotron Radiation
(spectrum)

$$\Delta t^A = t_2^A - t_1^A = \frac{2}{\gamma \omega_B \sin\alpha}\left(1 - \frac{v}{c}\right)$$

Since γ>>1 we have

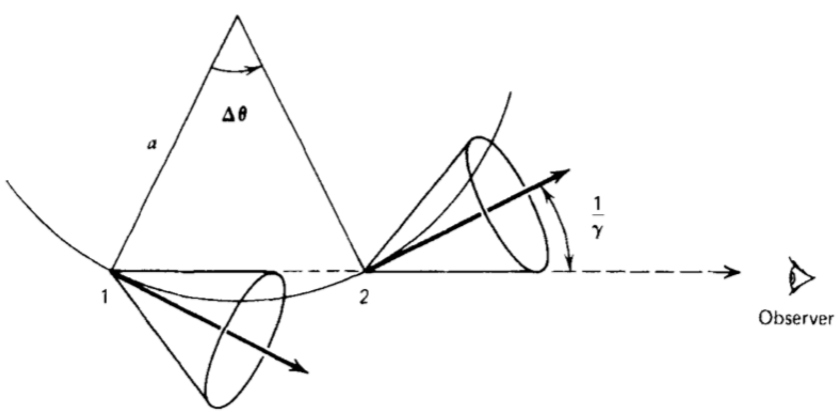
$$1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}$$

$$\Delta t^A \approx \left(\gamma^3 \omega_B \sin\alpha\right)^{-1}$$

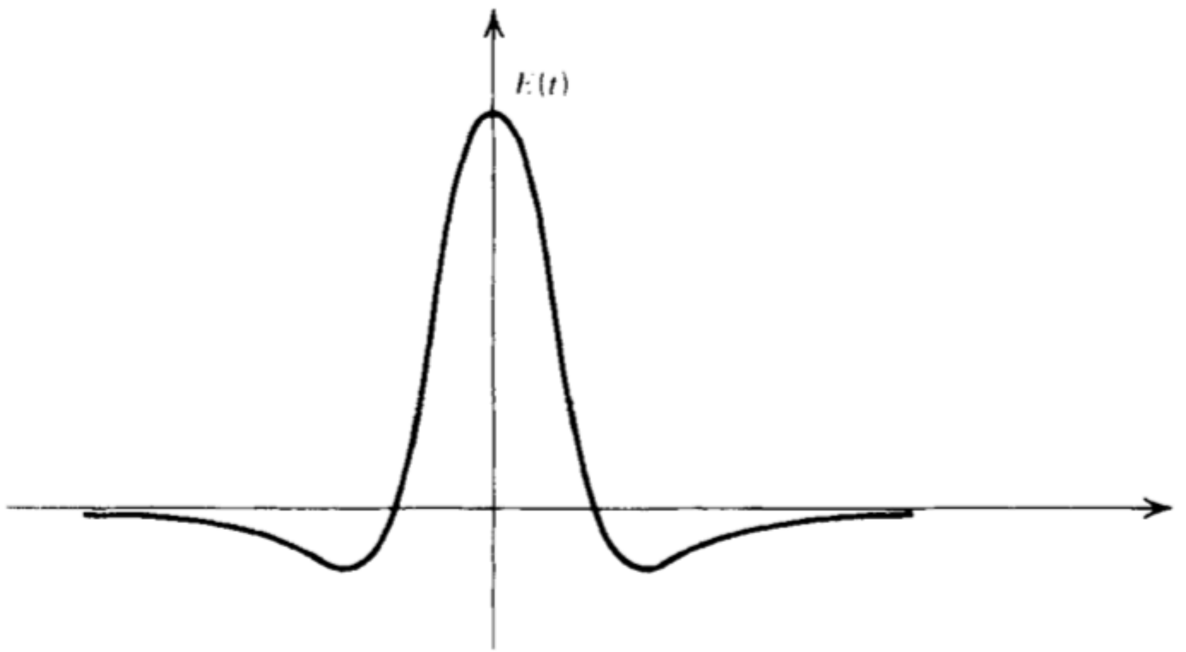
Width of observed pulses is smaller than gyration frequency by a factor of γ^3
So the spectrum will be broad with cutoff frequency $1/\Delta t^A$
Critical frequency:

$$\omega_c \equiv \frac{3}{2}\gamma^3 \omega_B \sin\alpha$$

$$\nu_c = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha,$$



Synchrotron Radiation
(spectrum)



Time-dependence of the electric field in a pulse of synchrotron radiation

Synchrotron Radiation
(spectrum)

Beaming effect

Electric field is function of γθ, where θ is polar angle about the direction of motion

$$E(t) \propto F(\gamma\theta)$$

t is time measured in observer's frame, zero of time (and path length s) when pulse is centered on observer.

θ ~s/a

t~(s/v)(1-v/c)

Relation between θ and t

$$\gamma\theta \approx 2\gamma\left(\gamma^2 \omega_B \sin\alpha\right) t \propto \omega_c t$$

$$E(t) \propto g(\omega_c t)$$

Synchrotron Radiation
(spectrum)

Electric field ⟶ $E(t) \propto g(\omega_c t)$

Fourier transform of Electric field ⟶ $\hat{E}(\omega) \propto \int_{-\infty}^{\infty} g(\omega_c t) e^{i\omega t} dt$

Changing variable of integration to $\xi \equiv \omega_c t$,

$$\hat{E}(\omega) \propto \int_{-\infty}^{\infty} g(\xi) e^{i\omega\xi/\omega_c} d\xi$$

Synchrotron Radiation
(spectrum)

$$\hat{E}(\omega) \propto \int_{-\infty}^{\infty} g(\xi) e^{i\omega\xi/\omega_c} d\xi$$

Spectrum $dW/d\omega d\Omega$ is proportional to the square of $E(\omega)$

Integrating over solid angle and dividing by orbital period

Time averaged power per unit frequency

$$\frac{dW}{dt\, d\omega} = T^{-1}\frac{dW}{d\omega} \equiv P(\omega) = C_1 F\left(\frac{\omega}{\omega_c}\right)$$

Constant of proportionality

Total power

$$P = \int_0^{\infty} P(\omega) d\omega = C_1 \int_0^{\infty} F\left(\frac{\omega}{\omega_c}\right) d\omega = \omega_c C_1 \int_0^{\infty} F(x)\, dx$$

Synchrotron Radiation
(spectrum)

Total power

$$P = \int_0^\infty P(\omega)d\omega = C_1 \int_0^\infty F\left(\frac{\omega}{\omega_c}\right)d\omega = \omega_c C_1 \int_0^\infty F(x)\,dx$$

Previous results

$$P = \frac{2q^4B^2\gamma^2\beta^2\sin^2\alpha}{3m^2c^3} \qquad \omega_c = \frac{3\gamma^2 qB\sin\alpha}{2mc}$$

For highly relativistic case, power per unit frequency emitted by each electron is

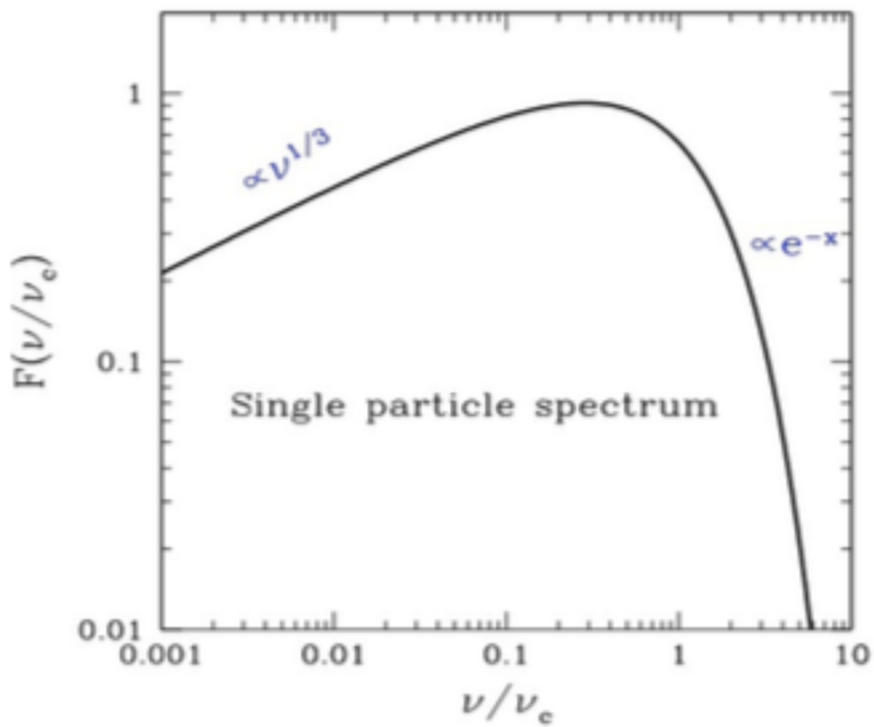
$$\boxed{P(\omega) = \frac{\sqrt{3}}{2\pi}\frac{q^3B\sin\alpha}{mc^2}F\left(\frac{\omega}{\omega_c}\right)}$$

Synchrotron Radiation
(single particle spectrum)

$$P(\omega)=\frac{\sqrt{3}}{2\pi}\frac{q^3 B\sin\alpha}{mc^2}F\left(\frac{\omega}{\omega_c}\right)$$

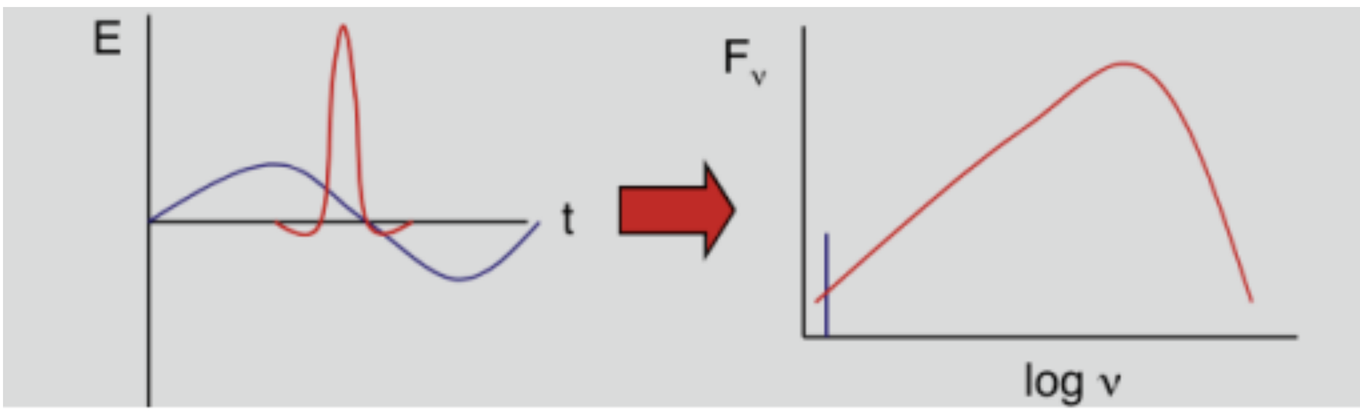
$$F(x)\sim\frac{4\pi}{\sqrt{3}\,\Gamma(\frac{1}{3})}\left(\frac{x}{2}\right)^{1/3},\qquad x\ll 1,$$

$$F(x)\sim\left(\frac{\pi}{2}\right)^{1/2}e^{-x}x^{1/2},\qquad x\gg 1.$$



Cylotron vs Synchrotron Radiation
(single particle spectrum)

Same physical origin but different spectra



Cyclotron spectra single line at

$\nu_L = qB/2\pi m$

Synchrotron spectrum

$$P(\omega) = \frac{\sqrt{3}}{2\pi} \frac{q^3 B \sin\alpha}{mc^2} F\left(\frac{\omega}{\omega_c}\right)$$

Synchrotron Radiation
(spectral index for power-law electron distribution)

$$P(\omega) = \frac{\sqrt{3}}{2\pi} \frac{q^3 B \sin\alpha}{mc^2} F\left(\frac{\omega}{\omega_c}\right)$$

No factor of γ in the formula other than in ω_c
The spectrum can be approximated by a power-law over a limited range of frequency. For that range let us imagine,

$$P(\omega) \propto \omega^{-s}.$$

Negative slope in P(ω) - log(ω) plot

Often the spectra of astronomical radiation has a spectral index that is constant over a fairly wide range of frequencies example s=-2 for Rayleigh-Jeans law

Synchrotron Radiation
(spectral index for power-law electron distribution)

Number density of particles with energies between E and E+dE

$$N(E)dE = CE^{-p}dE$$

Number density of particles with energies between γ and γ+dγ

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma$$

Total power radiated per unit volume per unit frequency is
N(γ)dγ times single particle radiation

$$P_{\text{tot}}(\omega) = C\int_{\gamma_1}^{\gamma_2} P(\omega)\gamma^{-p}d\gamma \propto \int_{\gamma_1}^{\gamma_2} F\left(\frac{\omega}{\omega_c}\right)\gamma^{-p}d\gamma$$

Synchrotron Radiation
(spectral index for power-law electron distribution)

Total power radiated per unit volume per unit frequency for an electron distribution

$$P_{tot}(\omega) = C\int_{\gamma_1}^{\gamma_2} P(\omega)\gamma^{-p}d\gamma \propto \int_{\gamma_1}^{\gamma_2} F\left(\frac{\omega}{\omega_c}\right)\gamma^{-p}d\gamma$$

$$\omega_c = \frac{3\gamma^2 qB\sin\alpha}{2mc}$$

Change variable of integration $x=\omega/\omega_c$

$$dx = -\frac{2\omega}{A\gamma^3}d\gamma \rightarrow \gamma^{-p}d\gamma = \frac{\gamma^{-p+3}A}{2\omega}dx = \left(\frac{\omega}{Ax}\right)^{(-p+3)/2}\frac{A}{2\omega}dx$$

$$P_{tot}(\omega) \propto \omega^{-(p-1)/2}\int_{x_1}^{x_2} F(x)x^{(p-3)/2}dx.$$

considering to be constant

$$P_{tot}(\omega) \propto \omega^{-(p-1)/2} \rightarrow s = \frac{p-1}{2}$$

Synchrotron Radiation
(spectral index for power-law electron distribution)

Total power radiated per unit volume per unit frequency
for an electron distribution (approximate calculation)

$$P_{\text{tot}}(\omega) \propto \omega^{-(p-1)/2} \int_{x_1}^{x_2} F(x) x^{(p-3)/2} dx.$$

Total power radiated per unit volume per unit frequency
for an electron distribution (detailed calculation)

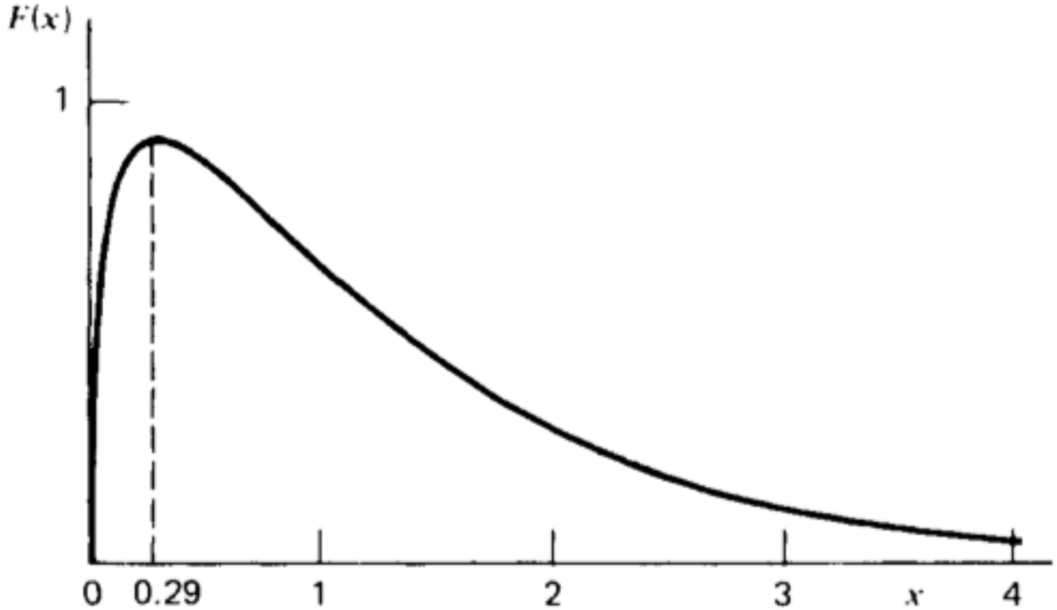
$$\frac{dW}{dt d\omega} \equiv P_\omega = \frac{\sqrt{3}}{2\pi} \frac{e^3 B \sin\alpha}{mc^2} F(\omega/\omega_c)$$

$$F(x) = x \int_x^\infty K_{5/3}(y) dy$$

For power law distribution of electrons,

$$P_{\text{tot}}(\omega) = \frac{\sqrt{3}\, q^3 C B \sin\alpha}{2\pi mc^2 (p+1)} \Gamma\left(\frac{p}{4} + \frac{19}{12}\right) \Gamma\left(\frac{p}{4} - \frac{1}{12}\right) \left(\frac{mc\omega}{3qB\sin\alpha}\right)^{-(p-1)/2}$$

Synchrotron Radiation
(spectral index for power-law electron distribution)



Synchrotron Radiation
(spectral index for power-law electron distribution)

- ✓ Angular distribution of single radiating particle is beamed ($1/\gamma$)

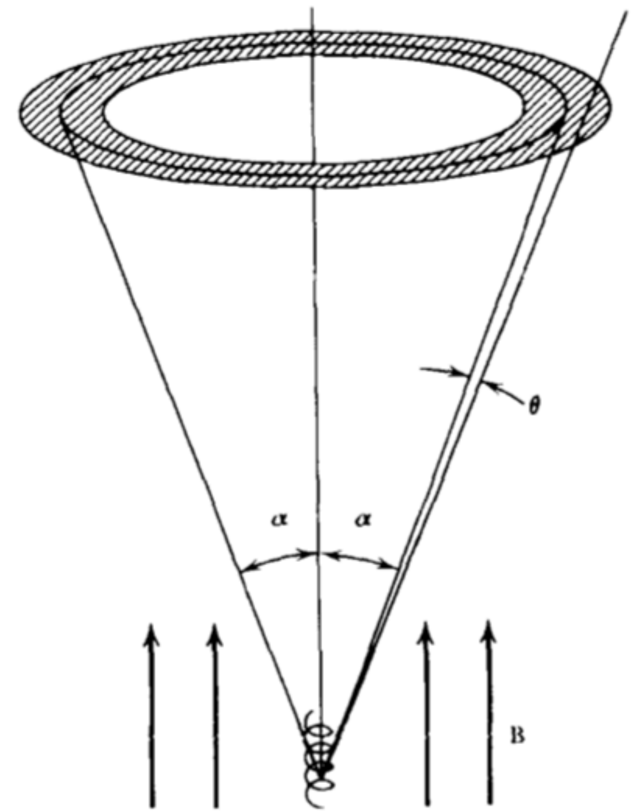
- ✓ Single particle spectrum extends up to $\sim\omega_c$

Spectrum function of ω/ω_c

- ✓ For multi particle system, power law distribution of energies with index p

Spectral index of radiation $s=(p-1)/2$

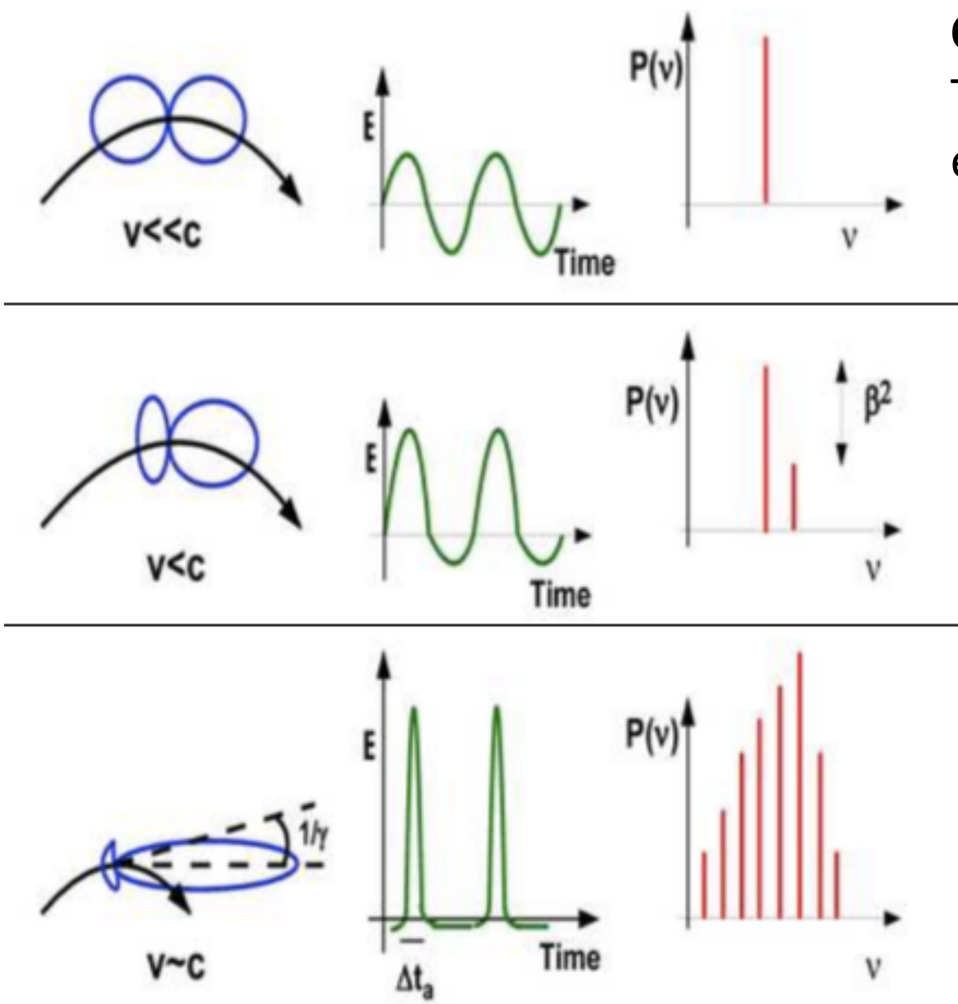
- ✓ Radiation is highly polarized



Synchrotron emission from a particle.
Radiation confined to the shaded region

Synchrotron Spectra
(transition from cyclotron to synchrotron emission)

Follow typical synchrotron spectrum as the electron's energy is varied from non-relativistic through highly relativistic regime.



Cyclotron radiation

The charge is moving in a circle, so the electric field variation is sinusoidal

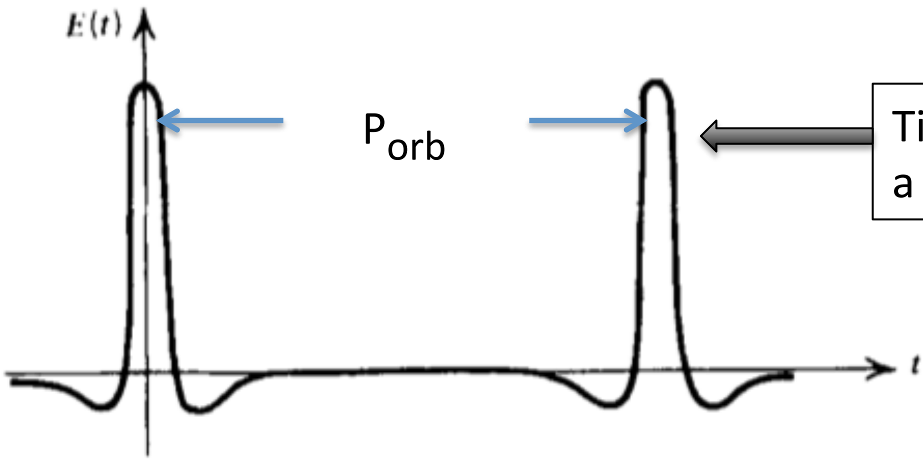
Cyclotron-synchrotron radiation

When v/c increases, higher harmonics of fundamental frequency ω_B begin to contribute

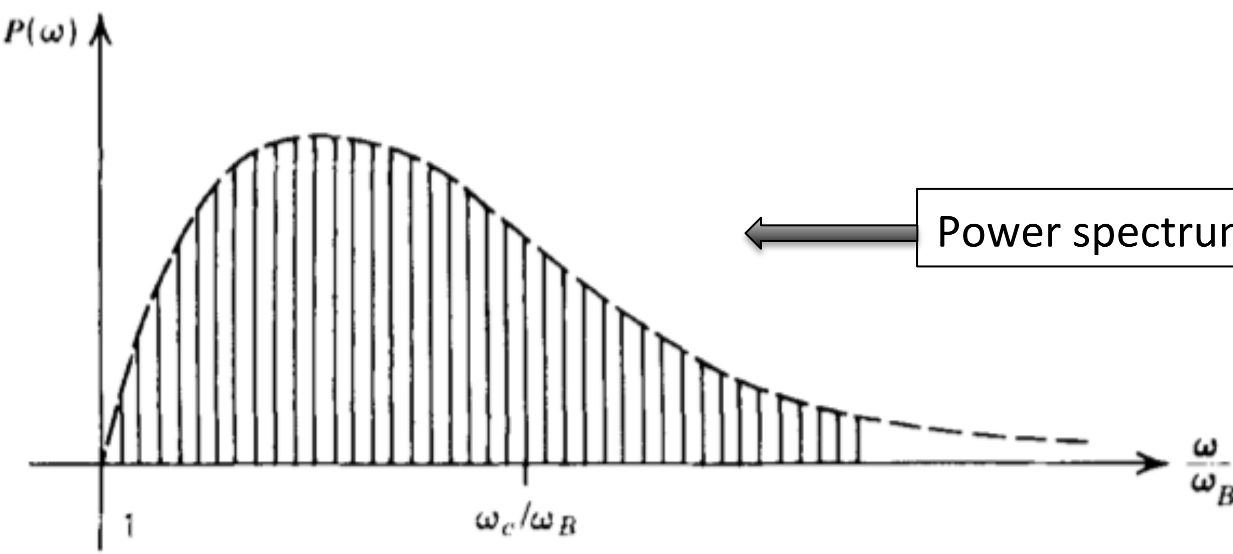
Synchrotron radiation

Charge is moving in a circle , and the radiation is seen only for a tiny amount of time when the cone $1/\gamma$ points towards the observers. Superposition of integral multiple of $\omega_{B.}$

Synchrotron Spectra



Time dependence of electric field from a rapidly moving particle in a magnetic field



Power spectrum

Synchrotron Spectra

For very relativistic velocities v~c, the originally sinusoidal form of E(t) has now become a series of sharp pulses which are repeated at time intervals $2\pi/\omega_{B.}$

The spectrum involves a large number of harmonics, the envelope of of which approaches F(x).

Why do we see continuous spectrum

a) As the frequency resolution becomes larger with respect to ω_B or other physical broadening mechanisms fills in the spaces between the lines (there is a distribution of particle with different energies and the gyration frequency ω_B is proportional to $1/\gamma \rightarrow$ the spectra of particles will not fall on the same lines.)

b) Emission from different parts of the emitting region may have different values and directions of the magnetic fields, so the harmonics fall at different places in the observed spectrum.

Synchrotron Spectra

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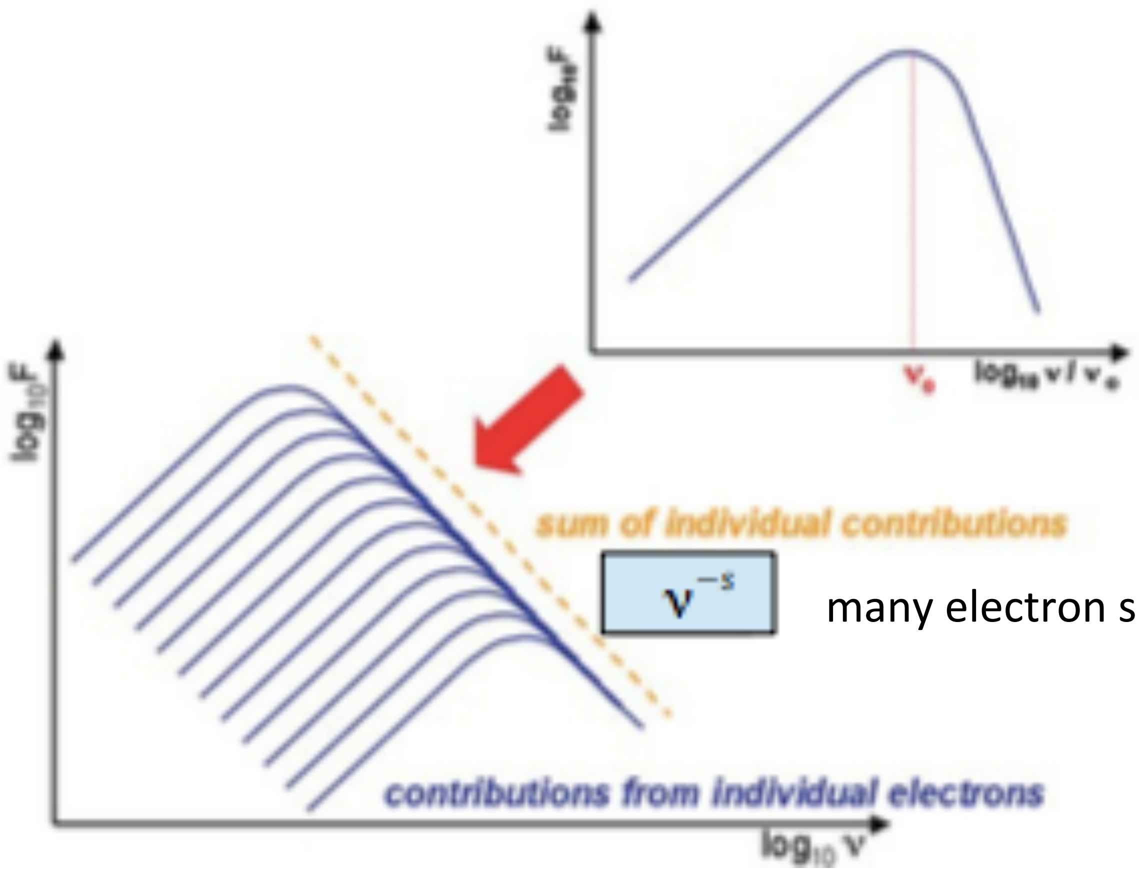
b) Emission from different parts of the emitting region may have different values and directions of the magnetic fields, so the harmonics fall at different places in the observed spectrum.

The electric field received by the observer from a distribution of particles consists of a random superposition of many pulses of the above kind. Net result is sum of spectra from individual pulses.

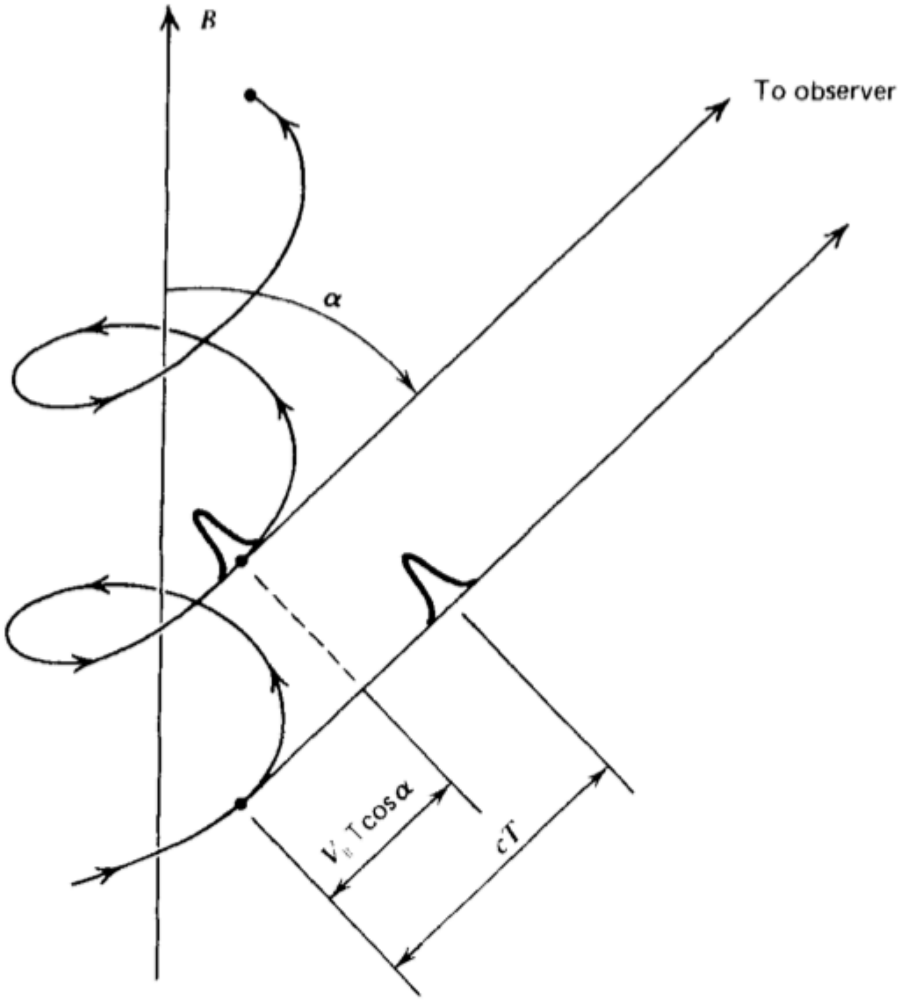
Synchrotron Spectra

Single electron spectra



many electron spectra

Distinction between received and emitted power



Doppler shift of synchrotron radiation emitted by a particle moving towards the observer

Received pulses are not at frequency ω_B but appropriately Doppler shifted because of progressive motion of particle towards observer.

If $T=2\pi/\omega_B$ is the orbital period of the projected motion, then time-delay effect will give a period between the arrival of pulses T_A

$$T_A = T\left(1 - \frac{v_\parallel}{c}\cos\alpha\right)$$

$$= T\left(1 - \frac{v}{c}\cos^2\alpha\right) \approx \frac{2\pi}{\omega_B}\sin^2\alpha.$$

Distinction between received and emitted power

The fundamental observed frequency is $\omega_B/\sin^2\alpha$

$$P_r = \frac{P_e}{\sin^2\alpha}.$$

For usual situation encountered in astrophysics one should use expression of emitted power to give observed power. Above correction due to helical motion are not important for most cases of interest.

Synchrotron cooling time

If we know the total emitted power we can calculate the cooling time of an ensemble of electrons emitting synchrotron.

$$t_{\rm syn} = \frac{E}{P} = \frac{\gamma m_{\rm e} c^2}{(4/3)\sigma_{\rm T} c U_B \gamma^2 \beta^2} \sim \frac{7.75 \times 10^8}{B^2 \gamma}\ {\rm s} = \frac{24.57}{B^2 \gamma}\ {\rm yr}$$

Example: Consider a supermassive black hole in an Active Galactic Nucleus.
The magnetic field around the black hole is of the order of 1,000 G
The Lorentz factor is also of the order of 1,000, so the electrons cool down on a timescale of just 0.77 seconds.

Polarization of Synchrotron Radiation

Radiation from single charge is elliptically polarized.

For a distribution of particles the radiation is partially linearly polarized.

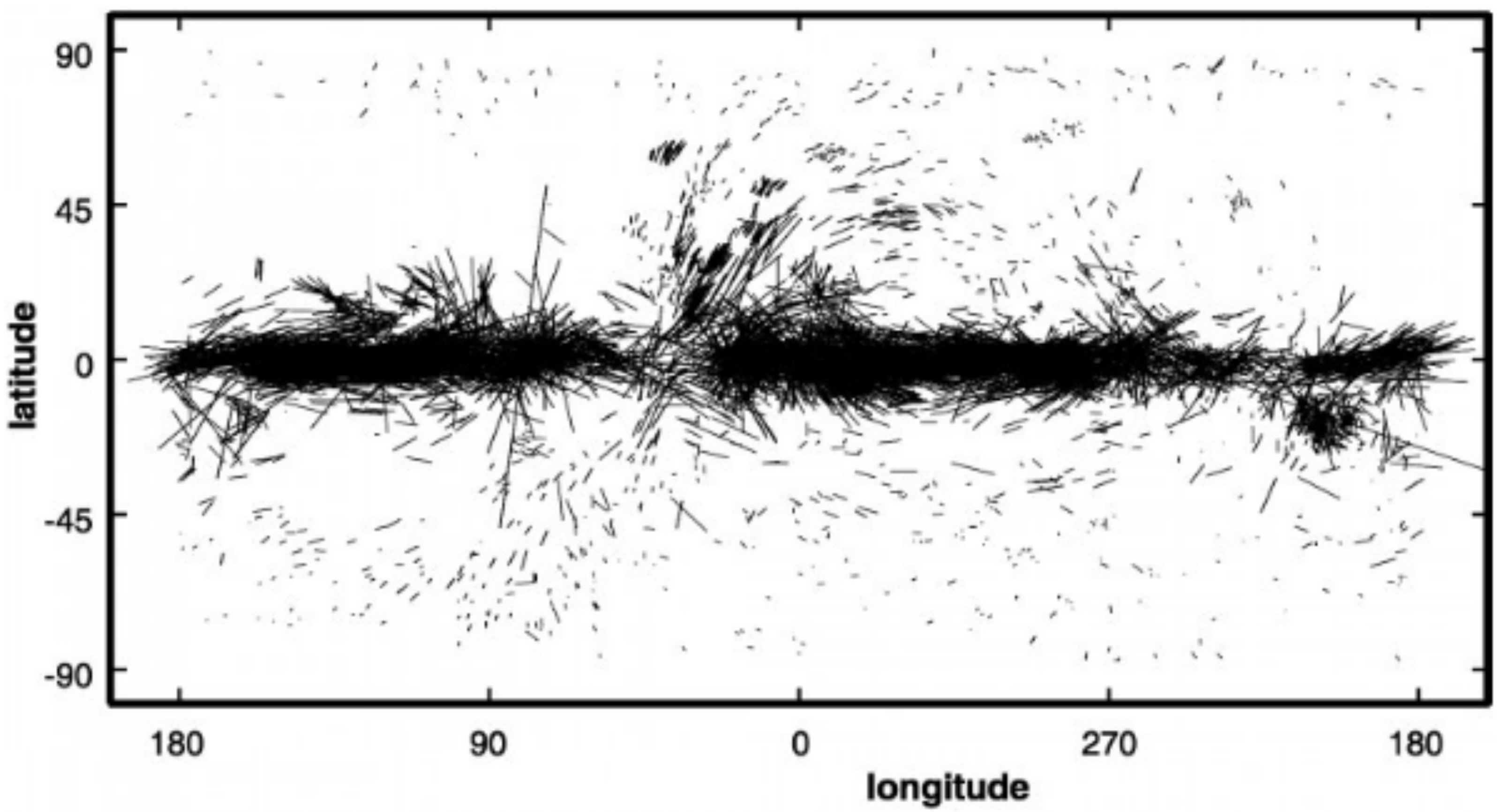
Polarization for frequency integrated radiation is 75% (Problem 6.5 in R&L)

For particles with power-law distribution of energy the degree of polarization,

$$\Pi = \frac{p+1}{p+\frac{7}{3}}.$$

=75% for p=3

Synchrotron in Astrophysics :
Large scale structure of Galactic Magnetic Field



Large scale map of the galactic magnetic field can be measured from polarization of radio emission coming from synchrotron processes. relativistic particles interact with the interstellar magnetic field and emit polarized synchrotron radiation

Synchrotron self-absorption

Synchrotron emission process is accompanied by absorption in which

a) A photon interacts with a charge in magnetic field and is absorbed giving up its energy to the charge

b) Stimulated emission (or negative absorption) in which a particle is induced to emit more strongly into a direction and at a frequency where photons are already present

These processes are related by Einstein's coefficient

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi_{21}(\nu)$$

$\varphi_{21}(\nu)$ is δ function that restricts summations to these states differing by an energy $h\nu = E_2 - E_1$

Synchrotron self-absorption

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} [\, n(E_1) B_{12} - n(E_2) B_{21}] \phi_{21}(\nu)$$

Now we want to write the absorption coefficient so that it contain the expression of power which we discussed,

$$P(\omega) = \frac{\sqrt{3}}{2\pi} \frac{q^3 B \sin\alpha}{mc^2} F\left(\frac{\omega}{\omega_c}\right)$$

It is convenient to write the emission in terms of the frequency ν rather than ω. So we use $P(\nu, E_2) = 2\pi P(\omega)$.

Synchrotron self-absorption

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} [n(E_1)B_{12} - n(E_2)B_{21}]\phi_{21}(\nu)$$

It is convenient to write the emission in terms of the frequency ν rather than ω. So we use $P(\nu, E_2) = 2\pi P(\omega)$.

Relations between Einstein's coefficients

$$g_1 B_{12} = g_2 B_{21},$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Total power emitted per frequency of a single particle can be written as

$$P(\nu, E_2) = h\nu \sum_{E_1} A_{21}\phi_{21}(\nu)$$

$$= (2h\nu^3/c^2) h\nu \sum_{E_1} B_{21}\phi_{21}(\nu)$$

This expression relates the spontaneous emission (A21) with the stimulated emission (B21)

Synchrotron self-absorption

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi_{21}(\nu)$$

Absorption coefficient due to stimulated emission

$$\frac{-h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_2) B_{21} \phi_{21} = \frac{-c^2}{8\pi h\nu^3} \sum_{E_2} n(E_2) P(\nu, E_2).$$

Absorption coefficient due to true absorption

$$\frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_1) B_{12} \phi_{21} = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} n(E_2 - h\nu) P(\nu, E_2)$$

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} \left[n(E_2 - h\nu) - n(E_2) \right] P(\nu, E_2)$$

Synchrotron self-absorption

Consider isotropic electron distribution function f(p)

f(p) d^3p =number of electrons per unit volume with momentum in d^3p about p

$$= \quad \tilde{\omega} h^{-3} d^3p,$$

(statistical weight of the particle, it has nothing to do with angular frequency; for electrons it's 2 (spin up/spin down states))

So we can make the substitution $\sum_2 \rightarrow \frac{\tilde{\omega}}{h^3} \int d^3p_2,$ $\quad n(E_2) \rightarrow \frac{h^3}{\tilde{\omega}} f(p_2).$

Synchrotron self-absorption

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} [n(E_2 - h\nu) - n(E_2)] P(\nu, E_2).$$

$+$

$$\sum_2 \to \frac{\tilde{\omega}}{h^3} \int d^3p_2,$$

$$n(E_2) \to \frac{h^3}{\tilde{\omega}} f(p_2).$$

So the absorption coefficient is

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int d^3p_2 [f(p_2^*) - f(p_2)] P(\nu, E_2)$$

where p_2^* is the momentum corresponding to energy E_2-hν

Check if this formula produces correct results for thermal distribution of particles

$$f(p) = K \exp\left[-\frac{E(p)}{kT}\right].$$

Synchrotron self-absorption

$$f(p_2^*) - f(p_2) = K \exp\left(-\frac{E_2 - h\nu}{kT}\right) - K \exp\left(-\frac{E_2}{kT}\right)$$

$$= f(p_2)(e^{h\nu/kT} - 1).$$

The absorption coefficient is

$$(\alpha_\nu)_{\text{thermal}} = \underbrace{\frac{c^2}{8\pi h\nu^3}(e^{h\nu/kT} - 1)}_{} \underbrace{\int d^3p_2 f(p_2) P(\nu, E_2)}_{}$$

$4\pi B_\nu(T)$

total power per unit volume per unit frequency range $4\pi j_\nu$

$$(\alpha_\nu)_{\text{thermal}} = \frac{j_\nu}{B_\nu(T)}$$

→ Correct result for thermal emission
Kirchhoff's Law

Synchrotron self-absorption

The last step is to consider the power law distribution of particles and get rid of f(p)

This will be done in next Lecture

End of Lecture 11

Next Lecture :18th September