Astronomical Techniques II
Lecture 14 - Sensitivity and a few misc. topics

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March-May 2015

Noise and Temperature

1. $P = k_B T \Delta\nu$
2. $P_a = g^2\ k_B T_a \Delta\nu$
3. $P_N = g^2\ k_B T_{sys} \Delta\nu$

   $T_{sys} = T_{leak} + T_{atm} + T_{spill} + T_{loss} + T_{rec} + T_{bg}$
   Everything but the target source
4. $P_a = \frac{1}{2}\ g^2\ \eta_a A\ S\ \Delta\nu = g^2\ k_B\ K\ S\ \Delta\nu$
5. $K = \frac{\eta_a A}{2k_B}$ $(K\ Jy^{-1})$ - Flux collecting ability of an antenna
6. System equivalent flux density - $SEFD = \frac{T_{sys}}{K}$

Sensitivity of a 2 element interferometer

1. $< P_i >= a_i < (s_i + n_i)^2 >= a_i[< s_i >^2 + < n_i >^2]$
2. $< P_i >= g_i^2 \ k_B \ (T_{a\ i} + T_{sys\ i}) \ \Delta\nu$
3. $< P_i >= g_i^2 \ k_B \ (K_i \ S_T + T_{sys\ i}) \ \Delta\nu$
4. $< P_{ij} >= \frac{g_i g_j}{\eta_s} \ \sqrt{K_i K_j} \ k_B \ \Delta\nu \ S_c$

Sensitivity of a 2 element interferometer

1. SNR - ratio of DC component to the RMS fluctuations of the correlator output
2. $\Delta S_{ij} =$
$$\frac{1}{\eta_s \sqrt{2 \, \Delta\nu \, \tau_{acc}}} \sqrt{S_c^2 + S_T^2 + S_T\left(\frac{T_{sys\ i}}{K_i} + \frac{T_{sys\ j}}{K_j}\right) + \frac{T_{sys\ i} T_{sys\ j}}{K_i K_j}}$$
    1. Assumes a square bandpass, but can be generalized to an arbitrary bandpass

Sensitivity of a 2 element interferometer

1. Weak source case $S_T << \frac{T_{sys}}{K}$

$$\Delta S_{ij} = \frac{1}{\eta_s}\sqrt{\frac{T_{sys\ i}T_{sys\ j}}{2\ \Delta\nu\ \tau_{acc}\ K_i K_j}} = \frac{1}{\eta_s}\sqrt{\frac{SEFD_i\ SEFD_j}{2\ \Delta\nu\ \tau_{acc}}}$$

2. Strong source case $S_T >> \frac{T_{sys}}{K}$

$$\Delta S_{ij} = \frac{S_T}{\eta_s\sqrt{2\ \Delta\nu\ \tau_{acc}}}$$

   1. Usually $S_T >> S_C$

Amplitudes and Phases

1. $S_m = \sqrt{S_R^2 + S_I^2}$

   $$\phi_m = tan^{-1}\frac{S_I}{S_R}$$

2. Noise distribution for S_m - Rice distribution

   $$P(S_m) = \frac{S_m}{\Delta S^2}\, I_0\left(\frac{S_m S}{\Delta S^2}\right) e^{\dfrac{-(S_m^2 + S^2)}{s\Delta S^2}}$$

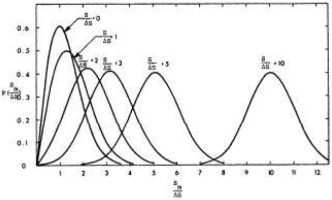
   where I_0 is the modified Bessels function of the first kind, order zero, and S is the true amplitude.

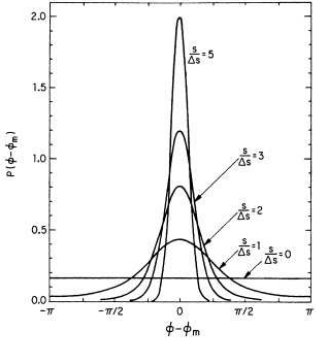
3. Probability distribution for phase error $\phi - \phi_m$, where ϕ is the true phase

   $$P(\phi - \phi_m) = \frac{1}{2\pi} e^{\dfrac{-S^2}{2\,\Delta S^2}} \left(1 + G\sqrt{\pi} e^{G^2}(1 + erfG\right)$$

   where $G(\theta) = \dfrac{Scos\theta}{\sqrt{2}\Delta S}$

Probability distribution of measured amplitude and phase





Sensitivity for a point source

1. $\Delta I_m = \dfrac{\sqrt{2} k_B T_{sys}}{\eta_c \eta_a A \sqrt{N_{base} \Delta\nu \; N_{times}}}$

2. $\eta_c = \dfrac{Sensitivity \; of \; the \; correlator}{Sensitivity \; of \; a \; perfect \; analog \; correlator}$

   1 bit - 64%; 2 bit 3 level - 81%

3. $\Delta I_m = \dfrac{1}{\eta_c} \dfrac{SEFD}{\sqrt{N(N-1) \; \Delta\nu \; N_{times}}}$

Effect of the primary beam

1. $I_m(l, m) = I(l, m)\ P(l, m) + N(l, m)$
2. $\dfrac{I_m(l, m)}{P(l, m)} = I(l, m) + \dfrac{N(l, m)}{P(l, m)}$

A formalism for 3-D imaging

1.

$$V(u,v,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathcal{A}(l,m)\, I(l,m)}{\sqrt{1-l^2-m^2}} e^{-2\pi i(ul+vm+w(\sqrt{1-l^2-m^2}-1))}\, dl\, dm$$

2.

$$V(u,v,w)\, e^{-2\pi i w} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathcal{A}(l,m)\, I(l,m)}{\sqrt{1-l^2-m^2}} \delta(n-\sqrt{1-l^2-m^2}) e^{-2\pi i(ul+vm+wn)}\, dl\, dm\, dn$$

1. $$I^{D(3)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(u,v,w)\ S(u,v,w) e^{-2\pi i w} e^{2\pi i(ul+vm+wn)} du\ dv\ dw$$

2. $I^{D(3)} = I^{(3)} \bigstar B^{D(3)}$ where,

$$I^{(3)}(l,m,n) = \frac{\mathcal{A}(l,m)\ I(l,m)}{\sqrt{1-l^2-m^2}}\ \delta(n - \sqrt{1-l^2-m^2})$$

3D Imaging

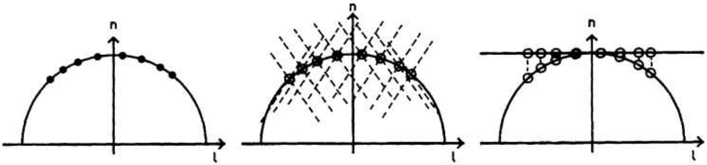


Figure 19–1. The image volume and its relation to the sky brightness. *(Left)* Three-dimensional transformation of the analytic visibility function maps the sky brightness onto a unit sphere. The dots represent these sources. *(Middle)* Convolution with a dirty beam results in sidelobes, shown as dashed lines, throughout the volume above and below the unit sphere. *(Right)* After deconvolution, the images are represented by finite-size "clean beams" on the unit sphere. The two-dimensional image is recovered by projection onto the tangent plane, indicated by vertical dashed lines.

3D Imaging

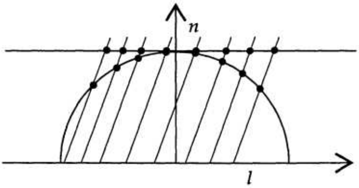


Figure 19–2. The image volume and its relation to a 'standard' two-dimensional image. *(Left)* At a particular time, a 'snapshot' with a two-dimensional array will project the true structure on the unit sphere onto the tangent plane with a 'ray beam', tilted at a particular angle given by the geometry of the array at the time of observation. *(Right)* At a later time, the array geometry has changed due to earth rotation, so the projection is now at a different angle. The apparent positions of the objects which are not located at the tangent point have changed with respect to the earlier observation.

3D Imaging

1. Faceting/Polyhedron imaging
   1. Divide the image into many many facets, each small enough that the small FoV and small w term approximation are sastified within it
   2. CLEAN flux is subtracted from ungridded visibilities
   3. No. of facets depends upon FoV and resolution
   4. 100-1000 times slower than 2D imaging
2. w projection (Cornwell, Golap and Bhatnagar, 2008)
   1. $V(u,v,w) =$
      $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{I(l,m)}{\sqrt{1-l^2-m^2}} \mathbf{G}(l,m,w) e^{-2\pi i(ul+vm)}\, dl\, dm,$$
      where
      $$\mathbf{G}(l,m,w) = e^{-2\pi i(w\sqrt{1-l^2-m^2}-1)}$$
   2. $V(u,v,w) = \hat{G}(u,v,w) \bigstar V(u,v,w=0)$
   3. Order of magnitude faster

Polarization

1. Most non-thermal processes give rise to at least partially polarised emission
2. Polarized emission is an important diagnostic of the conditions in the radio source and in the intervening medium
3.
4. Additional DoF needed to describe the polarization state of radiation
5. A given feed is sensitive to only one of the orthogonal pols (linear or circular)
6. Measure both polarizations and compute all four cross-correlations

Polarization Measurements

1. Significantly harder than total intensity
    1. For the vast majority of sources, fractional polarization is quite low - pushed into low SNR regimes
    2. Number of DoF for imaging increase by a factor of 4
    3. Calibration issues
        1. Instrumental
        2. Propagation
        3. Need for polarization calibrator
        4. Calibration tends to have a strong direction dependence (absolute, as well as within the fov)
        5. Alt-Az mounts

The Hamaker-Bregman-Sault Measurement Equation

1. Hamaker, Bregman and Sault - 1996-1998
2. Jones Matrix
   1. $E_0\ cos(\omega t + \phi) = E_0 e^{i\phi}$
   2. $\begin{pmatrix} E'_R \\ E'_L \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_R \\ E_L \end{pmatrix}$
   3. $J_{gain} = \begin{pmatrix} g_R & 0 \\ 0 & g_L \end{pmatrix}$
   4. $J_{leakage} = \begin{pmatrix} 1 & D_R \\ -D_L & 1 \end{pmatrix}$
   5. $J_{rotation} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$
   6. $J_{overall} = J_{gain}\ J_{leakage}\ J_{rotation} \cdots$

Jones matrices

1. Js are different for each antenna, and are usually time and frequency dependent
2. Provide a framework to represent propagation of signal path up to the correlator
3. Complicated systems can be handled gracefully
4. Provides an approach which allows individual effects to be modelled in different physically relevant manners
5. Matrix formulation - well suited for computational scalability and efficiency

Jone matrices - Polarimetric Equivalent

1. $V'_{i,j} = g_i \; g_j^* \; V_{i,j}$
2. $\mathbf{A} \bigotimes \mathbf{B} = a_{i,j}\mathbf{B}$

   $$(\mathbf{A}_i\mathbf{B}_i) \bigotimes (\mathbf{A}_j\mathbf{B}_j) = (\mathbf{A}_i \bigotimes \mathbf{A}_j)(\mathbf{B}_i \bigotimes \mathbf{B}_j)$$
3. Inputs to the correlator - $E'_i = \mathbf{J}_i \; E_i$
4. Outputs of the correlator - $E'_i \bigotimes E'^*_j$

   $$(\mathbf{J}_i E_i) \bigotimes (\mathbf{J}_j E_j)^* = (\mathbf{J}_i \bigotimes \mathbf{J}_j^*)(E_i \bigotimes E_j^*)$$
5. $E'_i \bigotimes E'^*_j = \begin{pmatrix} E_{R,i}E^*_{R,j} \\ E_{R,i}E^*_{L,j} \\ E_{L,i}E^*_{R,j} \\ E_{L,i}E^*_{L,j} \end{pmatrix}$

1. $< E'_i \bigotimes E'^*_j >= \left(\begin{array}{c} V_{RR,ij} \\ V_{RL,ij} \\ V_{LR,ij} \\ V_{LL,ij} \end{array} \right)$
2. $V'_{ij} = (\mathbf{J}_i \bigotimes \mathbf{J}^*_j) V_{ij}$ - V_{ij} - Coherency vector
3. Calibration requires estimating the different J_is and applying the inverse matrix to the measured Coherency vector

Relationship with Stokes vectors

1. $V_S = \begin{pmatrix} V_I \\ V_Q \\ V_U \\ V_V \end{pmatrix}$ - Stokes Visibility Vector
2. $V_{ij} = \mathbf{S}\ V_{S,ij}$
3. $V'_{ij} = (\mathbf{J}_i \bigotimes \mathbf{J}_j)\ \mathbf{S}\ V_{S,ij}$
4. $\mathbf{S}_{circ} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$ $\mathbf{S}_{linear} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \\ 1 & -1 & 0 & 0 \end{pmatrix}$