Bull. Astr. Soc. India (2011) **39**, 203–224

Gravitational waves from perturbed stars

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Received 2011 February 11; accepted 2011 February 28

Abstract. Non radial oscillations of neutron stars are associated with the emission of gravitational waves. The characteristic frequencies of these oscillations can be computed using the theory of stellar perturbations, and they are shown to carry detailed information on the internal structure of the emitting source. Moreover, they appear to be encoded in various radiative processes, as for instance in the tail of the giant flares of Soft Gamma Repeaters. Thus, their determination is central to the theory of stellar perturbation. A viable approach to the problem consists in formulating this theory as a problem of resonant scattering of gravitational waves incident on the potential barrier generated by the spacetime curvature. This approach discloses some unexpected correspondences between the theory of stellar perturbations and the theory of quantum mechanics, and allows us to predict new relativistic effects.

Keywords : gravitational waves – black hole physics – stars: oscillations – stars: neutron – stars: rotation

1. Introduction

The theory of stellar perturbations is a very powerful tool to investigate the features of gravitational signals emitted when a star is set in non-radial oscillations by any external or internal cause. The characteristic frequencies at which waves are emitted are of great interest in these days, since gravitational wave detectors Virgo and LIGO are approaching the sensitivity needed to detect gravitational waves emitted by pulsating stars. These frequencies carry information on the internal structure of a star, and appear to be encoded in various radiative processes; thus, their study is a central problem in perturbation theory and in astrophysics. In this paper I will illustrate the theory of perturbations of a non-rotating star, describing in particular the formulation

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that Chandrasekhar and I developed, its motivation and outcomes. Furthermore, I will briefly describe how the theory has been applied to study the oscillation frequencies of neutron stars.

In order to frame the problem in an appropriate historical perspective, it is instructive to remind ourselves how the study of stellar perturbations was treated in the framework of Newtonian gravity. In that case, the adiabatic perturbations of a spherical star are described by a fourth-order, linear, differential system which couples the perturbation of the Newtonian potential to those of the stellar fluid. All perturbed quantities are Fourier-expanded and, after a suitable expansion in spherical harmonics, which allows for the separation of variables, the relevant equations are manipulated in such a way that the quantity which is singled out to describe the perturbed star is the Lagrangian displacement $\vec{\xi}$ experienced by a generic fluid element; indeed, the changes in density, pressure and gravitational potential induced by the perturbation, can all be expressed uniquely in terms of $\vec{\xi}$. The equations for $\vec{\xi}$ have to be solved by imposing appropriate boundary conditions at the centre of the star, where all physical quantities must be regular, and on its boundary, where the perturbation of the pressure must vanish. These conditions are satisfied only for a discrete set of *real* values of the frequency, $\{\omega_n\}$, which are the frequencies of the star's *normal modes*. Thus, the linearized version of the Poisson and of the hydro equations are reduced to a characteristic value problem for the frequency ω.

An adequate base for a rigorous treatment of stellar pulsations of a spherical star in general relativity was provided by K.S. Thorne and collaborators in a series of papers published in the late sixties–early seventies of the last century (Thorne & Campolattaro 1967, 1968; Campolattaro & Thorne 1970; Thorne 1969a,b; Ipser & Thorne 1973). The theory was developed in analogy with the Newtonian approach, and was later completed by Lindblom & Detweiler (1983), who brought the analytic framework to a form suitable for the numerical integration of the equations, thus allowing for the determination of the real and imaginary part of the characteristic frequencies of the $\ell = 2$, *quasi-normal modes*. Indeed, a main difference between the Newtonian and the relativistic theory is that in general relativity the oscillations are damped by the emission of gravitational waves, and consequently the mode eigenfrequencies are complex. Higher order ($\ell > 2$) mode frequencies were subsequently computed by Cutler & Lindblom (1987).

In 1990, Professor Chandrasekhar and I started to work on stellar perturbations, and we decided to derive ab initio the equations of stellar perturbations following a different approach, having as a guide the theory of black hole perturbations rather than the Newtonian theory of stellar perturbations.

1.1 Black hole perturbations: wave equations and conservation laws

In 1957 T. Regge and J.A. Wheeler set the basis of the theory black hole perturbations showing that, by expanding the metric perturbations of a Schwarzschild black hole in tensorial spherical harmonics, Einstein's equations can be separated (Regge & Wheeler 1957). Spherical harmonics belong to two different classes, depending on the way they transform under the parity transformation $\theta \to \pi - \theta$ and $\varphi \to \pi + \varphi$; those which transform like $(-1)^{(\ell+1)}$ are named *odd*, or *axial*, those that transform like $(-1)^\ell$ are named *even*, or *polar*. The perturbed equations decouple in two distinct sets belonging to the two parities. Regge & Wheeler further showed that by Fourier-transforming the time dependent variables, the equations describing the radial part of the *axial* perturbations can easily be reduced to a single Schroedinger-like equation, and 13 years later Frank Zerilli showed that this can also be done for the much more complicated set of *polar* equations (Zerilli 1970a,b); thus, the axial and polar perturbations of a Schwarzschild black hole are described by the wave equation

$$\frac{d^2 Z_\ell^\pm}{dr_*^2} + \left[\omega^2 - V_\ell^\pm(r)\right] Z_\ell^\pm = 0\,, \tag{1}$$

$$V_\ell^-(r) = \frac{1}{r^3}\left(1 - \frac{2M}{r}\right)[\ell(\ell+1)r - 6M] \tag{2}$$

$$V_\ell^+(r) = \frac{2(r-2M)}{r^4(nr+3M)^2}[n^2(n+1)r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3]\,. \tag{3}$$

where $r_* = r + 2M\log\left(\frac{r}{2M} - 1\right)$, $n = \frac{1}{2}(\ell+1)(\ell-2)$, and M is the black hole mass. The superscript $-$ and $+$ indicate, respectively, the Regge-Wheeler equation for the axial perturbations, and the Zerilli equation for the polar perturbations, and the corresponding potentials. The Regge-Wheeler potential for $\ell = 2$ is shown in figure 1.

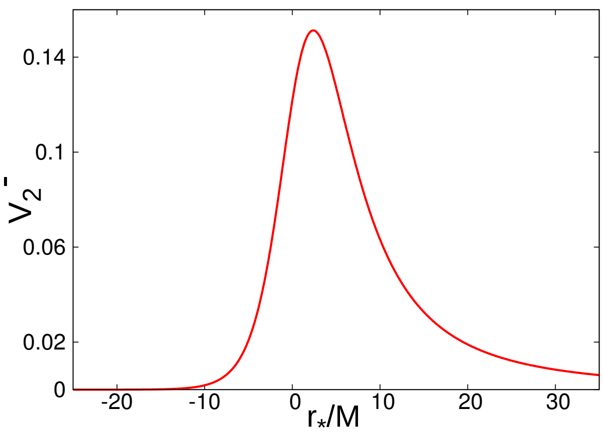

Figure 1. The potential barrier generated by the axial perturbations of a Schwarzschild black hole for $\ell = 2$. The potential for the polar perturbations has a similar form.

An alternative approach to studying black hole perturbations considers the perturbations of the Weyl and Maxwell scalars within the Newman-Penrose formalism. Using this approach in 1972 S. Teukolsky was able to decouple and separate the equations governing the perturbations of a Kerr black hole (Teukolsky 1972, 1973), and to reduce them to a single master equation for the radial part of the perturbation R_{lm}:

$$\Delta R_{lm,rr} + 2(s+1)(r-M)R_{lm,r} + V(\omega,r)R_{lm} = 0\ , \qquad \Delta = r^2 - 2Mr + a^2. \tag{4}$$

Variable separation was achieved in terms of oblate spheroidal harmonics, and the potential $V(\omega, r)$ is given by

$$V(\omega,r) = \frac{1}{\Delta}\left[(r^2+a^2)^2\omega^2 - 4aMrm\omega + a^2m^2 + 2is(am(r-M) - M\omega(r^2-a^2))\right] \tag{5}$$

$$+\left[2is\omega r - a^2\omega^2 - A_{lm}\right]\ , \tag{6}$$

where a is the black hole angular momentum, A_{lm} is a separation constant, and s, the spin-weight parameter, takes the values $s = 0, \pm1, \pm2$, respectively for scalar, electromagnetic and gravitational perturbations. It may be noted that for the Schwarzschild perturbations, due to the background spherical symmetry, non-axisymmetric modes with a $e^{im\phi}$ dependence can be deduced from axisymmetric, $m = 0$ modes by suitable rotations of the polar axes. As a consequence, the potentials (2) and (3) do not depend on the harmonic index m. Conversely, since Kerr's background is axisymmetric, this degeneracy is removed and the potential (6) depends on m. Moreover, while the Schwarzschild potentials (2) and (3) are real and independent of frequency, the potential barrier of a Kerr black hole is complex, and depends on frequency.

The wave equations governing black hole perturbations show that the curvature generated by a black hole appears in the perturbed equations as a one-dimensional potential barrier; consequently, the response of a black hole to a generic perturbation can be studied by investigating the manner in which a gravitational wave incident on that barrier is transmitted, absorbed and reflected. Thus, the theory of black hole perturbations can be formulated as a scattering theory, and the methods traditionally applied in quantum mechanics to investigate the behaviour of physical systems described by a Schroedinger equation can be adapted and used to study the behaviour of perturbed black holes. For instance, it is known that in quantum mechanics, given a one-dimensional potential barrier associated with a Schroedinger equation, the singularities in the scattering cross-section correspond to complex eigenvalues of the energy and to the so-called quasi-stationary states. Since the perturbations of a Schwarzschild black hole are described by the Schroedinger-like equation (1) with the one-dimensional potential barriers (2) and (3), in which the energy is replaced by the frequency, the singularities in the scattering cross-section will provide the complex values of the black hole eigenfrequencies, and the corresponding eigenstates will be the black hole Quasi Nomal Modes (QNM). These modes satisfy the boundary conditions of a pure outgoing gravitational wave emerging at radial infinity ($r_* \to +\infty$), and a pure ingoing wave impinging at the black hole horizon ($r_* \to -\infty$). That these solutions should exist had been suggested by C.V. Vishveshwara in 1970 (Vishveshwara 1970), and the next year W.H. Press confirmed this idea by numerically integrating the axial wave equation (1), and by showing that an arbitrary initial perturbation ends in a ringing tail, which indicates that black holes possess some proper modes of vibration (Press 1971). However, it was only in 1975 that S. Chandrasekhar and S. Detweiler computed the complex eigenfrequencies of the quasi-normal modes of a Schwarzschild black hole, by integrating the Riccati equation associated with the axial equation (1) (Chandrasekhar & Detweiler 1975). In addition, they also showed that the transmission and the reflection coefficients associated respectively with the polar and with the axial potential barriers are equal. As a consequence, the polar and the axial perturbations are isospectral, i.e. the polar and axial QNM eigenfrequencies are equal. This equality can be explained in terms of a transformation theory which clarifies the relations that exist between potential barriers admitting the same reflection and absorption coefficients. This is an example of how the scattering approach has been effective not only to determine the QNM frequencies, but also to investigate the inner relations existing among the axial and polar potential barriers and to gain a deeper insight in the mathematical theory of black holes, which was illustrated by S. Chandrasekhar in his book on the subject (Chandrasekhar 1984). Following this approach, a variety of methods developed in the context of quantum mechanics have been used to determine the QNM spectra of rotating and non-rotating black holes, such as the WKB and higher order WKB method, phase-integral methods and the theory of Regge poles, just to mention some of them (Schutz & Will 1985; Ferrari & Mashhoon 1984a,b; Andersson, Araujo & Schutz 1993a,b,c; Andersson 1994; Andersson & Thylwe 1994).

1.2 A conservation law for black hole perturbations and its generalization to perturbed stars

In quantum mechanics the equation

$$|R|^2 + |T|^2 = 1, \tag{7}$$

where R and T are the reflection and transmission coefficients associated with a potential barrier, expresses the symmetry and unitarity of the scattering matrix; it says that, if a wave of unitary amplitude is incident on one side of the potential barrier, it gives rise to a reflected and a transmitted wave such that the sum of the square of their amplitudes is still one. Therefore, equation (7) is an energy conservation law for the scattering problem described by the Schroedinger equation with a potential barrier. This conservation law is a consequence of the constancy of the Wronskian of pairs of independent solutions of the Schroedinger equation. Similarly, the constancy of the Wronskian of two independent solutions of the black hole wave equations allows us to write the same relation between the reflection and transmission coefficients associated with the potential barrier, and therefore it shows that such an energy conservation law also governs the scattering of gravitational waves by a perturbed black hole. It should be stressed that such energy conservation law *does not* exist in the framework of the exact non-linear theory; however, it can be derived in perturbation theory both for Schwarzschild, Kerr and Reissner-Nordstrom black holes.

This possibility led Chandrasekhar to the following consideration. Since in general relativity, any distribution of matter (or more generally energy of any sort) induces a curvature of the spacetime – a potential well – instead of picturing the non-radial oscillations of a star as caused by some unspecified external perturbation, we can picture them as excited by incident gravitational radiation. Viewed in this manner, the reflection and absorption of incident gravitational waves by black holes and the non-radial oscillations of stars, become different aspects of the same basic theory. However, this idea needed to be substantiated by facts, and our starting point was to show that also for perturbed stars it is possible to write an energy conservation law in terms of Wronskians of independent solutions of the perturbation equations of a spherical star. This is easy if we consider the axial perturbations of a non-rotating star, because in that case, as we shall show in Section 2, the perturbed equations can be reduced to a wave equation with a one dimensional potential barrier as for black holes. However, to derive the conservation law for the polar perturbations was not easy, because the corresponding equations are a fourth order linear differential system, in which the perturbed metric functions couple to the fluid perturbations, and it was not clear how to define the conserved current. Anyway, working hard on the equations, we were able to derive a vector $\vec{\mathbf{E}}$ in terms of metric and fluid perturbations, which satisfies the following equation (Chandrasekhar & Ferrari 1990a):

$$\frac{\partial}{\partial x^\alpha} E^\alpha = 0, \qquad \alpha = (x^2 = r, x^3 = \vartheta). \tag{8}$$

(It is worth reminding ourselves that, due to the spherical symmetry, it is not restrictive to consider axisymmetric perturbations $m = 0$). The vanishing of the ordinary divergence implies that, by Gauss's theorem, the flux of $\vec{\mathbf{E}}$ across a closed surface surrounding the star is a constant. When the fluid variables are switched off, this conservation law reduces to that derived for a Schwarzschild black hole, and therefore we thought we were on the right track. However, there was still a question to answer: are we entitled to say that the vector $\vec{\mathbf{E}}$ actually represents the flux of gravitational energy which develops through the stars and propagates outside? If so, equation (8) should reduce to the second variation of the time component of the well known equation

$$\frac{\partial}{\partial x^\nu}[\sqrt{-g}(T^{\mu\nu} + t^{\mu\nu})] = 0. \tag{9}$$

where $t^{\mu\nu}$ is the stress-energy pseudotensor of the gravitational field. The problem is that $t^{\mu\nu}$ is not uniquely defined; indeed equation (9) shows that it is defined up to a divergenceless term. A possible definition is that given by Landau & Lifschitz (1975), which has the advantage of being symmetric. However, the second variation of the time component of equation (9) assuming $t^{\mu\nu} = t^{\mu\nu}_{LL}$, does not give the divergenceless equation satisfied by our vector $\vec{\mathbf{E}}$, neither for the Einstein-Maxwell case, nor in the case of a star. Then, Raphael Sorkin suggested that the pseudo-tensor whose second variation should reproduce our conserved current is the Einstein pseudo-tensor, because its second variation retains its divergence-free property, provided only the equations governing the static spacetime and its linear perturbations are satisfied[1]. This property is a consequence of the Einstein pseudo-tensor being a Noether operator for the gravitational field; the Landau-Lifshitz pseudotensor failed to reproduce the conserved current because it does not satisfy the foregoing requirements. In addition, Sorkin pointed out that the contribution of the source should be introduced not by adding the second variation of the source stress-energy tensor $T^{\mu\nu}$, as one might naively have thought, but through a suitably defined Noether operator, whose form he derived for an electromagnetic field (Sorkin 1991). Though this operator does not coincide with $T^{\mu\nu}$, it gives the same conserved quantities. Thus, the flux integral which we had obtained, I would say, by brute force, working directly on the perturbed hydrodynamical equations, could be obtained from a suitable expansion of the Einstein pseudo-tensor showing that,

[1]It should be mentioned that the first variation of the Einstein pseudo-tensor vanishes identically.

as for black holes, energy conservation also governs phenomena involving gravitational waves emitted by perturbed stars (Chandrasekhar & Ferrari 1991a). We therefore decided to derive ab initio the equations of perturbations of a spherical star in the same gauge used when studying the perturbations of a Schwarzschild black hole, and to study the problem as a scattering problem. In the next sections I shall briefly illustrate the main results we obtained by using this approach (Chandrasekhar & Ferrari 1990b, 1991b,c, 1992; Chandrasekhar, Ferrari & Winston 1991).

2. Perturbations of a non-rotating star

As for a Schwarzschild black hole, when the equations describing the perturbations of a spherical star are perturbed and expanded in spherical tensor harmonics they decouple in two distinct sets, one for the polar and one for the axial perturbations. The axial equations do not involve fluid motion except for a stationary rotation, while the polar equations couple fluid and metric perturbations. The axial equations are therefore much simpler and we showed that, after separating the variables and Fourier-expanding the perturbed functions, they can be combined as in the Schwarzschild case, and reduced to a single Schroedinger-like equation with a one-dimensional potential barrier (Chandrasekhar & Ferrari 1990b, 1991c):

$$\frac{d^2 Z_\ell^-}{dr_*^2} + [\omega^2 - V_\ell^-(r)]Z_\ell^- = 0, \tag{10}$$

where

$$r_* = \int_0^r e^{-\nu+\mu_2} dr, \tag{11}$$

and

$$V_\ell^-(r) = \frac{e^{2\nu}}{r^3}[\ell(\ell+1)r + r^3(\epsilon - p) - 6m(r)]. \tag{12}$$

The functions $\nu(r)$ and $\mu_2(r)$, which appear in the definition of the radial variable r_*, are two metric functions which are found by solving the equations of stellar structure for an assigned equation of state (EOS). $\epsilon(r)$ and $p(r)$ are the energy density and the pressure in the unperturbed star; outside the star they vanish and equation (12) reduces to the Regge-Wheeler potential (2) of a Schwarzschild black hole. Thus, the axial potential barrier generated by the curvature of the star depends on how the energy-density and the pressure are distributed inside the star in the equilibrium configuration, and therefore it depends on the equation of state of matter inside the star. As an example, in Figure 2 we show the $\ell = 2$ potential barrier for an ideal, constant density star with $R/M = 2.8$ (left panel) and $R/M = 2.4$ (right panel). If we compare the potential shown in Figure 2 with the Regge-Wheeler potential of a Schwarzschild black hole shown in Figure 1, we notice an important difference. The Schwarzschild potential vanishes at the black hole horizon, and has a maximum at $r_{max} \sim 3M$, whereas the potential barrier of a perturbed star tends to infinity at $r = 0$. Thus, for a Schwarzschild black hole waves are scattered by a one-dimensional potential barrier, whereas in the case of a star they are scattered by a central potential.

Since the axial perturbations do not excite any motion in the fluid, for a long time they have

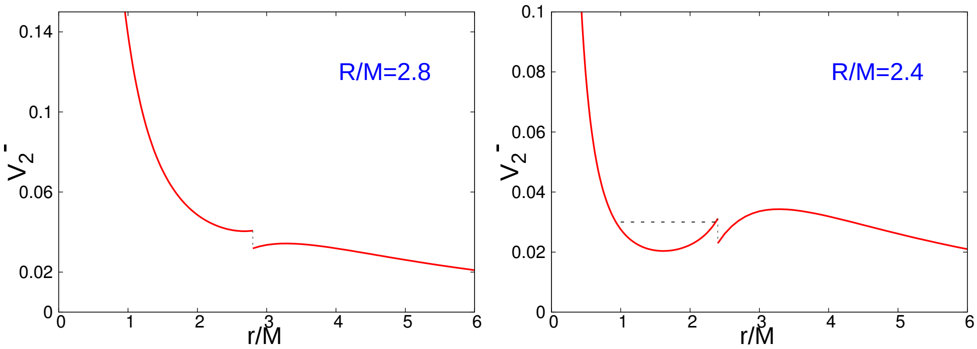

Figure 2. The $\ell = 2$ potential barrier for a constant density star of mass $M = 1.4\ M_\odot$ and $R/M = 2.8$ (left panel) and $R/M = 2.4$ (right panel). The horizontal, dashed line in the right panel corresponds to the value of frequency ω^2, which corresponds to a solution regular at $r = 0$ and behaving as a pure outgoing wave at radial infinity, i.e. to a quasi-normal mode.

been considered as trivial. But this is not true if we adopt the scattering approach: the absence of fluid motion simply means that the incident axial wave experiences a potential scattering, and this scattering can, in some extreme conditions, be resonant. Indeed, if we look for solutions that are regular at $r = 0$ and behave as pure outgoing waves at infinity, we find modes which do not exist in Newtonian theory; if the star is extremely compact, the potential in the interior is a well, and if this well is deep enough there can exist one or more more slowly damped quasi-normal modes, or *s*-modes (Chandrasekhar & Ferrari 1991c). For example, if the mass of the star is, say, $M = 1.4\ M_\odot$ and $R/M = 2.4$, i.e. the stellar compactness is $M/R = 0.42$, as shown in Figure 2 the well inside the star is deep enough to allow one quasi-normal mode. The number of *s*-modes increases with the depth of the well, which corresponds to a larger stellar compactness. However, it should be mentioned that neutron stars are not expected to have such a large compactness, unless one invokes some exotic equation of state. The *s*-modes are also named *trapped modes* because, due to the slow damping, they are effectively trapped by the potential barrier, and not much radiation can leak out of the star when these modes are excited. Axial modes on a second branch are named *w*-modes and are highly damped (Kokkotas 1994). The *w*-mode frequency also depends on the stellar compactness, as we shall show in Section 3.1. Therefore they carry interesting information on the internal structure of the star.

It should be stressed that the axial modes do not have a Newtonian counterpart.

Our approach to the polar perturbations, which couple the perturbations of the gravitational field to those of the metric, is different from the Newtonian approach briefly described in the introduction. Rather than focusing on the fluid behaviour, we focus on the variables which describe the spacetime perturbations, assuming that, as in the case of black holes, they are excited by the incidence of polar gravitational waves belonging to a particular angular harmonic. A careful scrutiny of the structure of the polar equations shows that it is possible to decouple the equations describing the metric from those describing the fluid perturbations. This decoupling allows us to solve the equations for the spacetime perturbations with no reference to the motion that can be induced in the fluid, and this is possible in general. Once the solution for the metric perturbations is found, the fluid variables can be determined in terms of them by simple algebraic relations without further ado (Chandrasekhar & Ferrari 1990b). The final set of equations to solve is described in Section 2.1.

2.1 The equations for the polar perturbations

Assuming that the metric which describes the unperturbed star has the form

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}d\varphi - e^{2\mu_2}(dr)^2 - e^{2\mu_3}(d\theta)^2, \tag{13}$$

the functions that describe the polar perturbations, expanded in spherical tensor harmonics and Fourier-expanded are

$$\begin{aligned} \delta\nu &= N_\ell(r)P_\ell(\cos\theta)e^{i\omega t} & \delta\mu_2 &= L_\ell(r)P_\ell(\cos\theta)e^{i\omega t} \\ \delta\mu_3 &= [T_\ell(r)P_\ell + V_\ell(r)P_{\ell,\theta,\theta}]e^{i\omega t} & \delta\psi &= [T_\ell(r)P_\ell + V_\ell(r)P_{\ell,\theta}\cot\theta]e^{i\omega t}, \\ \delta p &= \Pi_\ell(r)P_\ell(\cos\theta)e^{i\omega t} & 2(\epsilon+p)e^{\nu+\mu_2}\xi_r(r,\theta)e^{i\omega t} &= U_\ell(r)P_\ell e^{i\omega t} \\ \delta\epsilon &= E_\ell(r)P_\ell(\cos\theta)e^{i\omega t} & 2(\epsilon+p)e^{\nu+\mu_3}\xi_\theta(r,\theta)e^{i\omega t} &= W_\ell(r)P_{\ell,\theta}e^{i\omega t}, \end{aligned} \tag{14}$$

where $P_\ell(\cos\theta)$ are Legendre's polynomials, ω is the frequency, δp and $\delta\epsilon$ are perturbations of the pressure and of the energy density, and ξ_r, ξ_θ are the relevant components of the Lagrangian displacement of the generic fluid element. Note that (N, L, T, V) and (Π, E, U, W) are, respectively, the radial part of the metric and of the fluid perturbations. After separating the variables the relevant Einstein's equations for the metric functions become

$$\begin{cases} X_{\ell,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right)X_{\ell,r} + \frac{n}{r^2}e^{2\mu_2}(N_\ell + L_\ell) + \omega^2 e^{2(\mu_2-\nu)}X_\ell = 0, \\ (r^2G_\ell)_{,r} = n\nu_{,r}(N_\ell - L_\ell) + \frac{n}{r}(e^{2\mu_2} - 1)(N_\ell + L_\ell) + r(\nu_{,r} - \mu_{2,r})X_{\ell,r} + \omega^2 e^{2(\mu_2-\nu)}rX_\ell, \\ -\nu_{,r}N_{\ell,r} = -G_\ell + \nu_{,r}[X_{\ell,r} + \nu_{,r}(N_\ell - L_\ell)] + \frac{1}{r^2}(e^{2\mu_2} - 1)(N_\ell - rX_{\ell,r} - r^2G_\ell) \\ -e^{2\mu_2}(\epsilon + p)N_\ell + \frac{1}{2}\omega^2 e^{2(\mu_2-\nu)}\left\{N_\ell + L_\ell + \frac{r^2}{n}G_\ell + \frac{1}{n}[rX_{\ell,r} + (2n+1)X_\ell]\right\}, \\ L_{\ell,r}(1 - D) + L_\ell\left[\left(\frac{2}{r} - \nu_{,r}\right) - \left(\frac{1}{r} + \nu_{,r}\right)D\right] + X_{\ell,r} + X_\ell\left(\frac{1}{r} - \nu_{,r}\right) + DN_{\ell,r} + \\ N_\ell\left(D\nu_{,r} - \frac{D}{r} - F\right) + \left(\frac{1}{r} + E\nu_{,r}\right)\left[N_\ell - L_\ell + \frac{r^2}{n}G_\ell + \frac{1}{n}\left(rX_{\ell,r} + X_\ell\right)\right] = 0, \end{cases} \tag{15}$$

where

$$\begin{cases} A = \frac{1}{2}\omega^2 e^{-2\nu}, \qquad Q = \frac{(\epsilon+p)}{\gamma p}, \\ \gamma = \frac{(\epsilon+p)}{p}\left(\frac{\partial p}{\partial \epsilon}\right)_{entropy=const}, \qquad B = \frac{e^{-2\mu_2}\nu_{,r}}{2(\epsilon+p)}(\epsilon_{,r} - Qp_{,r}), \\ D = 1 - \frac{A}{2(A+B)} = 1 - \frac{\omega^2 e^{-2\nu}(\epsilon+p)}{\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \\ E = D(Q-1) - Q, \\ F = \frac{\epsilon_{,r} - Qp_{,r}}{2(A+B)} = \frac{2\left[\epsilon_{,r} - Qp_{,r}\right](\epsilon+p)}{2\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \end{cases} \tag{16}$$

and V_ℓ and T_ℓ have been replaced by X_ℓ and G_ℓ defined as

$$\begin{cases} X_\ell = nV_\ell \\ G_\ell = \nu_{,r}[\frac{n+1}{n}X_\ell - T_\ell]_{,r} + \frac{1}{r^2}(e^{2\mu_2} - 1)[n(N_\ell + T_\ell) + N_\ell] \\ + \frac{\nu_{,r}}{r}(N_\ell + L_\ell) - e^{2\mu_2}(\epsilon + p)N_\ell + \frac{1}{2}\omega^2 e^{2(\mu_2-\nu)}[L_\ell - T_\ell + \frac{2n+1}{n}X_\ell]. \end{cases} \tag{17}$$

These equations are valid in general, also for non-barotropic equations of state. It should be stressed that equations (15) govern the variables (X, G, N, L) which are *metric perturbations*; however, since the motion of the fluid is excited by the polar perturbation, we may want to determine the fluid variables, (Π, E, U, W); they can be obtained in terms of the metric functions using the following algebraic relations

$$W_\ell = T_\ell - V_\ell + L_\ell,$$

$$\Pi_\ell = -\frac{1}{2}\omega^2 e^{-2\nu} W_\ell - (\epsilon + p)N_\ell, \qquad E_\ell = Q\Pi_\ell + \frac{e^{-2\mu_2}}{2(\epsilon + p)}(\epsilon_{,r} - Qp_{,r})U_\ell,$$

$$U_\ell = \frac{[(\omega^2 e^{-2\nu} W_\ell)_{,r} + (Q+1)\nu_{,r}(\omega^2 e^{-2\nu} W_\ell) + 2(\epsilon_{,r} - Qp_{,r})N_\ell](\epsilon + p)}{[\omega^2 e^{-2\nu}(\epsilon + p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})]}.$$

Outside the star the fluid variables vanish, and the polar equations reduce to the wave equation (1) with the Zerilli potential (3).

As discussed above, for the axial perturbations, the frequencies of the quasi-normal modes were found by solving a problem of scattering by a central potential; for the polar perturbations it is not so simple, because a Schroedinger equation holds only in the exterior of the star, whereas a higher order system must be solved in the interior. It is still a scattering problem, but of a more complex nature since the incident polar gravitational waves, which excite the perturbations, drive the fluid pulsations, which in turn emit the scattered component of the wave. This approach was very fruitful in many respects. First of all, given the equilibrium configuration for any assigned equation of state, it was very easy to evaluate the QNM-frequency by integrating the equations for the metric perturbations inside and outside the star, looking for the solutions which, being regular at $r = 0$, behave as pure outgoing waves at infinity. Furthermore, we generalized the perturbed equations to slowly rotating stars, and derived the equations which describe how the axial perturbations couple to the polar (Chandrasekhar & Ferrari 1991b).

2.2 Perturbed equations for a slowly rotating star

Very briefly, the coupling mechanism is the following. Let Z_ℓ^{0-} be the axial radial function, solution of equation (10), which describes the perturbation of a non-rotating star; let $\epsilon(\Omega)Z_\ell^{1-}$ be the perturbation to first order in the star angular velocity Ω. The axial perturbation is the sum of the two:

$$Z_\ell^- = Z_\ell^{0-} + \epsilon(\Omega)Z_\ell^{1-}.$$

As Z_ℓ^{0-}, the function Z_ℓ^{1-} satisfies the wave equation (10) with the same potential (12), but with a forcing term:

$$\sum_{\ell=2}^{\infty}\left\{\frac{d^2 Z_\ell^{-1}}{dr_*^2} + \left[\omega^2 - V_\ell^-\right]Z_\ell^{-1}\right\}C_{\ell+2}^{-\frac{3}{2}}(\mu) = re^{2\nu-2\mu_2}(1-\mu^2)^2\sum_{\ell=2}^{\infty}S_\ell^0(r,\mu), \tag{18}$$

where $\mu = \cos\vartheta$ and $C_{\ell+2}^{-\frac{3}{2}}(\mu)$ are the Gegenbauer polynomials. The source term S_ℓ^0 is

$$S_\ell^0 = \varpi_{,r}[(2W_\ell^0 + N_\ell^0 + 5L_\ell^0 + 2nV_\ell^0 P_{\ell,\mu} + 2\mu V_\ell^0 P_{\ell,\mu,\mu}] + 2\varpi W_\ell^0(Q-1)\nu_{,r}P_{\ell,\mu};$$

it is a combination of the functions which describe the *polar* perturbations on the *non-rotating* star, found by solving the equations given in Section 2.1. It should be stressed that the coupling function ϖ is the function responsible for the Lense-Thirring effect. Thus a rotating star exerts a dragging not only of the bodies, but also of the waves, and consequently an incoming polar gravitational wave can convert, through the fluid oscillations it excites, some of its energy into outgoing axial waves. This is a purely relativistic effect, and it is due to the dragging of inertial frames. It is interesting to note that the coupling between axial and polar perturbations satisfies rules that are similar to those known in the theory of atomic transitions: a Laporte rule and a selection rule, according to which the polar modes belonging to *even* ℓ can couple only with the axial modes belonging to *odd* ℓ, and conversely, and that it must be

$$l = m + 1, \qquad \text{or} \qquad l = m - 1.$$

Furthermore, the coupling satisfies a propensity rule (Fano 1985): the transition $\ell \to \ell + 1$ is strongly favoured over the transition $\ell \to \ell - 1$.

At the time Chandrasekhar and I wrote the series of papers on stellar perturbations, there was a growing interest in the subject, also motivated by the fact that the construction of ground based interferometric detectors, LIGO in the US and Virgo in Italy, had just started. Many studies addressed the problem of finding the frequencies of the QNMs, to establish what kind of information they carry on the internal structure of the emitting source. The collective effort developed using essentially two different perturbative approaches: one in the frequency domain, as for the theory developed by Thorne and collaborators or by Chandrasekhar and myself, another in the time domain. The time domain approach basically consists in separating the equations of stellar perturbations as usual in terms of spherical harmonics, and in solving the resulting equations in terms of two independent variables, radial distance and time. The equations are excited using some numerical input, like for instance a Gaussian impulse, and then the QNM frequencies are found by looking at the peaks of the Fourier transform of the signal obtained by evolving the time-dependent equations numerically. A disadvantage of this evolution scheme is that one cannot get the complete spectrum of the QNMs either for a star or for a black hole. The reason is that, although any perturbation is the sum of the harmonics involved, in practice only a few of them can be clearly identified; thus, to find some more modes one has to proceed empirically by changing the initial conditions. However, the evolution of the time dependent equations is, still today, the only viable perturbative method to find the QNM frequencies and waveforms emitted by rapidly rotating relativistic stars. To describe the problems which emerge when dealing with the perturbations of a rapidly rotating star is beyond the scope of this paper; I will discuss some related issues in the concluding remarks.

3. Neutron star oscillations

We shall now show how the theory of perturbations of non-rotating stars can be applied to gain some insight into the internal structure of the emitting source. Different classes of modes probe different aspects of the physics of neutron stars. For instance the fundamental mode (f-mode), which has been shown to be the most efficient GW emitter by most numerical simulations, depends on the average density, the pressure modes (p-modes) probe the sound speed throughout the star, the gravity modes (g-modes) are associated with thermal/composition gradients and the w-modes are spacetime oscillations. Furthermore, crustal modes, superfluid modes, magnetic field modes can, if present, add to the complexity of stellar dynamics. The sensitivity of ground based gravitational detectors has steadily improved over the years in a broad frequency window; the advanced version of LIGO and Virgo, and especially third generation detectors like ET, promise to be powerful instruments to detect signals emitted by oscillating stars. The frequencies of quasi normal modes are encoded in these signals; therefore, as the Sun oscillation frequencies are used in helioseismology to probe its internal structure, we hope that in the future it will be possible to use gravitational waves to probe the physics of neutron stars. One of the issues which is interesting to address concerns the equation of state of matter in a neutron star core, which is actually unknown. This problem is of particular interest, because the energies prevailing in the inner core of a neutron star are much larger than those accessible to high energy experiments on Earth. In the core, densities typically exceed the equilibrium density of nuclear matter, $\rho_0 = 2.67 \times 10^{14}$ g/cm^3; at these densities neutrons cannot be treated as non-interacting particles, and the main contribution to pressure, which comes from neutrons, cannot be derived only from Pauli's exclusion principle. Indeed, with only this contribution, we would find that the maximum mass of a neutron star is 0.7 $M_\odot$ which, as observations show, is far too low. This clearly shows that NS equilibrium requires a pressure other than the degeneracy pressure, the origin of which has to be traced back to the nature of hadronic interactions. Due to the complexity of the fundamental theory of strong interactions, the equations of state appropriate to describe a NS core have been obtained within models, which are constrained, as much as possible, by empirical data. They are derived within two main, different approaches: the nonrelativistic nuclear many-body theory, NMBT, and the relativistic mean field theory, RMFT. In NMBT, nuclear matter is viewed as a collection of pointlike protons and neutrons, whose dynamics is described by the nonrelativistic Hamiltonian:

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{j>i} v_{ij} + \sum_{k>j>i} V_{ijk} \,, \tag{19}$$

where m and p_i denote the nucleon mass and momentum, respectively, whereas v_{ij} and V_{ijk} describe two- and three-nucleon interactions. These potentials are obtained from fits of existing scattering data (Wiringa, Stoks & Schiavilla 1995), (Pudliner et al. 1995). The ground state energy is calculated using either variational techniques or G-matrix perturbation theory. The RMFT is based on the formalism of relativistic quantum field theory, nucleons are described as Dirac particles interacting through meson exchange. In the simplest implementation of this approach the dynamics is modeled in terms of a scalar and a vector field (Walecka 1974). The equations of motion are solved in the mean field approximation, i.e. replacing the meson fields with their vacuum expectation values, and the parameters of the Lagrangian density, i.e. the meson masses and coupling constants, can be determined by fitting the empirical properties of nuclear matter, i.e. binding energy, equilibrium density and compressibility. Both NMBT and RMFT can be generalized to take into account the appearance of hyperons. In the following we shall consider some EOS representative of the two approaches, which have been used in the literature.

It should be stressed that different ways of modeling hadronic interactions affect the pulsation properties of a star, which we are going to discuss.

3.1 The axial and polar *w*-modes

As shown in Section 2, the axial perturbations are described by a Schroedinger-like equation with a central potential barrier which depends on the energy and pressure distribution in the unperturbed star, i.e. on the equation of state. The slowly damped modes are not expected to be associated with significant gravitational wave emission, because they are effectively trapped by the potential barrier; in addition they appear if the star has a compactness close to the static Schwarzschild limit, which establishes that constant density star solutions of Einstein's equations exists only for $M/R < 4/9 \simeq 0.44$. Conversely, the *w*-modes, which are highly damped, exist also for stars with ordinary compactness. They have been shown to exist also for the polar perturbations and in that case they are coupled to negligible fluid motion. In Figure 3 we compare the frequencies of the lower axial *w*-modes computed in (Benhar, Berti & Ferrari 1999) with those of the lower polar *w*-modes computed in (Andersson & Kokkotas 1988) for several EOSs. The main features of different EOS are, very briefly, the following. EOS A (Pandharipande 1971a) is pure neutron matter, with dynamics governed by a nonrelativistic Hamiltonian containing a semi-phenomenological interaction potential. It is obtained using NMBT. EOS B (Pandharipande 1971b) is a generalization of EOS A, including protons, electrons and muons in β-equilibrium, as well as heavier baryons (hyperons and nucleon resonances) at sufficiently high densities (NMBT). EOS WFF (Wiringa, Fiks & Fabrocini 1988) is a mixture of neutrons, protons, electrons and muons in β-equilibrium. The Hamiltonian includes two- and three-body interaction potentials. The ground state energy is computed using a more sophisticated and accurate many-body technique (NMBT). In EOS L (Pandharipande & Smith 1975) neutrons interact through exchange of mesons (ω, ρ, σ). The exchange of heavy particles (ω, ρ) is described in terms of nonrelativistic potentials, the effect of σ-meson is described using relativistic field theory and the mean-field approximation.

From Figure 3 we see that for each selected EOS the frequency of the *polar* w-modes is a rather steeply decreasing function of the stellar compactness M/R, whereas for the *axial* modes the dependence of ν_{w_0} on the compactness is weak, and ranges within intervals that are separated for each EOS. This means that if an axial gravitational wave emitted by a star at a given frequency could be detected, we would be able to identify the equation of state prevailing in the star's interior even without knowing its mass and radius. Hence, the detection of axial gravitational waves would allow us to constrain the EOS models, with regard to both the composition of neutron star matter and the description of the hadronic interactions. Until very recently, the common belief was that *w*- modes are unlikely to be excited in astrophysical processes. However, it has been shown that they are excited in the collapse of a neutron star to a black hole, just before the black hole forms (Baiotti et al. 2005). Unfortunately the typical frequencies of these modes (of the order of several kHz) are higher than the frequency region where the actual gravitational wave detectors are sensitive.

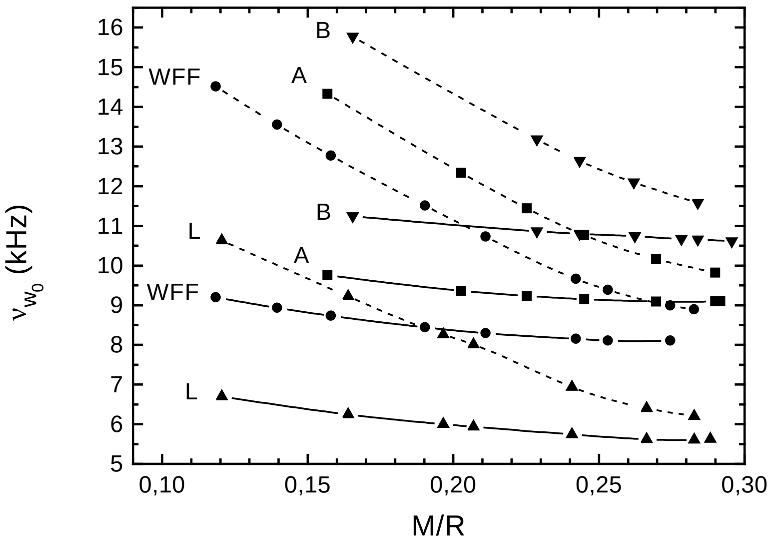

Figure 3. The frequency of the first polar (dashed line) and axial (continuous line) w-modes are plotted as a function of the star compactness for the EOSs A, B, WFF, L.

3.2 Polar Quasi Normal Modes

The polar metric perturbations are physically coupled to the fluid perturbations. As shown in Section 2.1, the frequencies of the polar QNMs can be computed by solving a system of equations involving only the metric perturbations; however, they carry a strong imprint of the internal composition of the star, which is present in equations (15) through the pressure and energy density profiles in the unperturbed star, which appear as coefficients of the differential equations. According to a scheme introduced by Cowling in Newtonian gravity (Cowling 1942), polar modes can be classified on the basis of the restoring force which prevails when the generic fluid element is displaced from the equilibrium position: for g-modes, or gravity modes, the restoring force is due to buoyancy, for p-modes it is due to pressure gradients. The mode frequencies are ordered as follows

$$..\omega_{g_n} < .. < \omega_{g_1} < \omega_f < \omega_{p_1} < .. < \omega_{p_n}..$$

and are separated by the frequency of the fundamental mode (f-mode), which has an intermediate character between $g-$ and $p-$ modes. As discussed in Section 3.1, general relativity predicts also the existence of polar w-modes, that are very weakly coupled to fluid motion and are similar to the axial w-modes (Kokkotas & Schutz 1992). The frequencies of axial and polar w-modes are typically higher than those of the fluid modes g, f and p.

If we are mainly interested in gravitational wave emission, the most interesting mode is the f-mode. For mature neutron stars, its frequency is in the range $1-3$ kHz, which is in the bandwidth of ground based detectors Virgo and LIGO (although not in the region where they are most sensitive); the damping times are of the order of a few tenths of seconds, therefore the excitation of the f-mode would appear in the Fourier transform of a gravitational wave signal as a sharp peak and could, in principle, be extracted from the detector noise by an appropriate data analysis. Moreover, the fundamental mode could be excited in several astrophysical processes, for instance in the aftermath of a gravitational collapse, in a glitch, or due to matter accretion onto the star. For this reason, since the early years of the theory of stellar perturbations, the interest of scientists working in this field has initially been focussed on the determination of the f-mode frequencies. After the work of Lindblom & Detweiler in 1983 and of Cutler & Lindblom in 1987, who respectively computed the $\ell = 2$ and $\ell > 2$ f-mode eigenfrequencies for the EOSs available at that time, more recently this work has been updated, and extended to other modes, by Anderson & Kokkotas (1998) and Benhar, Ferrari & Gualtieri (2004). In particular, in these two papers the f-mode frequency ν_f and the corresponding damping time τ_f have been computed to establish whether ν_f scales with the average density of the star, as it does in Newtonian gravity, and whether there also exists a scaling law for τ_f. The sets of EOSs used in the two works are not identical, because the papers were written six years apart, although some EOSs appear in both (see the two papers for details). The work done by Benhar, Ferrari & Gualtieri (2004) also includes examples of hybrid stars, namely neutron stars with a core composed of quarks. In Anderson & Kokkotas (1998) ν_f and τ_f have been fitted by a linear function of the average density of the star $(M/R^3)^{1/2}$, and of its compactness M/R, as follows.

$$\nu_f = 0.78 + 1.635\sqrt{\frac{\tilde{M}}{\tilde{R}^3}}, \qquad \frac{1}{\tau_f} = \frac{\tilde{M}^3}{\tilde{R}^4}\left[22.85 - 16.65\left(\frac{\tilde{M}}{\tilde{R}}\right)\right], \tag{20}$$

where $\tilde{M} = M/1.4\ M_\odot$ and $\tilde{R} = R/(10\text{ km})$. Here and in the following formulae ν_f is expressed in kHz and τ_f in s. The fits for ν_f and τ_f obtained by Benhar, Ferrari & Gualtieri (2004) using the new set of EOSs are

$$\nu_f = a + b\sqrt{\frac{M}{R^3}}, \quad a = 0.79 \pm 0.09 \text{ (in kHz)}, \quad b = 33 \pm 2 \text{ (in km)}, \tag{21}$$

and

$$\frac{1}{\tau_f} = \frac{cM^3}{R^4}\left[a + b\left(\frac{M}{R}\right)\right], \qquad a = [8.7 \pm 0.2] \cdot 10^{-2}, \qquad b = -0.271 \pm 0.009\,. \tag{22}$$

In equations (21) and (22) mass and radius are in km (i.e. mass is multiplied by G/c^2) and $c = 3 \cdot 10^5$ km/s. The data for the different EOSs used by Benhar, Ferrari & Gualtieri (2004), and the fits given in equations (20)-(22) are shown in Figure 4. ν_f is plotted in the upper panel as a function of the average density; the fit (20) is shown as a black dashed line labelled 'AK fit', whereas the new fit is indicated as a red continuous line labelled 'NS fit'. The NS fit is lower by about 100 Hz than the AK fit, showing that the new EOSs are, on average, less compressible than the old ones. The quantity $(R^4/cM^3)/\tau_f$ given in equation (22) is plotted in the lower panel of Figure 4 versus the stellar compactness M/R. In this case the AK fit for τ_f (20), and the NS fit (22) are nearly coincident. For comparison, in both panels of Figure 4 we show the frequency and the damping time of the f-mode of a population of strange stars, namely stars entirely made of up, down and strange quarks, modeled using the MIT Bag-model, spanning the allowed range

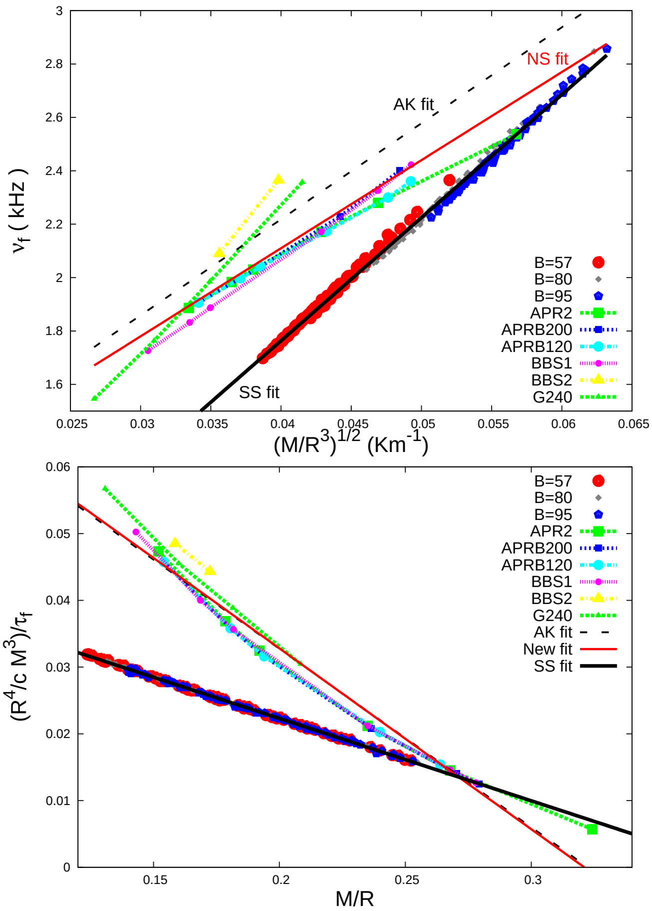

Figure 4. The frequency of the fundamental mode is plotted in the upper panel as a function of the square root of the average density for the different EOSs considered by Benhar, Ferrari & Gualtieri (2004). We also plot the fit given by Anderson & Kokkotas (1998) plotted as AK-fit and our fit (NS-fit). The NS-fit is systematically lower (about 100 Hz) than the AK-fit. The damping time of the fundamental mode is plotted in the lower panel as a function of the compactness M/R. The AK-fit and our fit, plotted respectively as a dashed and continuous line, do not show significant differences.

of parameters, which are the Bag constant, the coupling constant α_S and the quark masses (see Benhar et al. 2007 for details). The parameters of the fits for strange stars are

$$\text{for } \nu_f \qquad a = -[0.8 \pm 0.08] \cdot 10^{-2}\,, \qquad b = 46 \pm 0.2, \tag{23}$$

and

$$\text{for } \tau_f \qquad a = [4.7 \pm 5 \cdot 10^{-3}] \cdot 10^{-2}, \qquad b = -0.12 \pm 3 \cdot 10^{-4}\,. \tag{24}$$

In Figure 4 the fits for strange stars are labelled as 'SS fit'. It is interesting to note that the SS fits are quite different from those appropriate for neutron stars (AK- and NS-fits). First of all the errors on the parameters are much smaller, which indicate that the linear behaviour is followed by these stars, both for ν_f and for τ_f, irrespective of the values of the parameters of the model. Moreover, the difference between the fits is much larger for lower values of the average density.

The empirical relations given in eqs. (20)-(24) could be used to constrain the values of the star mass and radius, were the values of ν_f and τ_f identified in a detected gravitational signal. The stellar parameters would be further constrained if other modes are excited and detected and, knowing them, we would gain information on the equations of state of matter in the neutron star core, whose uncertainty is due, as explained earlier, to our ignorance of hadronic interactions. Furthermore, if the neutron star mass is known, as it may be if the star is in a binary system, the detection of a signal emitted by the star oscillating in the f-mode may provide some further interesting information (Benhar et al. 2007). In Figure 5 we plot ν_f as a function of the stellar mass, for neutron/hybrid stars and for strange stars modeled using the MIT bag model. Note that 1.8 $M_\odot$ is the maximum mass above which no stable strange star can exist. We see that there is a small range of frequency where neutron/hybrid stars are indistinguishable from strange stars; conversely, there is a large frequency region where only strange stars can emit. Moreover, strange stars cannot emit gravitational waves with $\nu_f \lesssim 1.7$ kHz, for any value of the mass in the range we consider. For instance, if the stellar mass is $M = 1.4\ M_\odot$, a signal with $\nu_f \gtrsim 2$ kHz would belong to a strange star. Figure 5 also shows that, even if we do not know the mass of the star (as it is often the case for isolated pulsars), if $\nu_f \gtrsim 2.2$ kHz, apart from a very narrow region of masses where stars with hyperons would emit (EOS BBS1 and G240), we can reasonably rule out that the signal is emitted by a neutron star. In addition, it is possible to show that, since ν_f is an increasing function of the Bag constant B, if a signal emitted by an oscillating strange star were detected, it would be possible to set constraints on B much more stringent than those provided by the available experimental data (Benhar et al. 2007).

In conclusion, the QNM frequencies can be used to gain direct information on the equation of state of matter in a neutron star core.

The crucial question now is: do we have a chance to detect a signal emitted by a star oscillating in a polar quasi-normal mode? Detection chances depend on how much energy is channeled into the pulsating mode, which is unknown, and on whether the mode frequency is in the detector bandwidth. The signal emitted by a star pulsating in a given mode of frequency ν and damping time τ, has the form of a damped sinusoid

$$h(t) = \mathcal{A} e^{-(t-t_0)/\tau} \sin[2\pi\nu(t - t_0)] \quad \text{for } t > t_0, \tag{25}$$

where t_0 is the arrival time of the signal at the detector (and $h(t) = 0$ for $t < t_0$). The wave amplitude $\mathcal{A}$ can be expressed in terms of the energy radiated in the oscillations,

$$\mathcal{A} \approx 7.6 \times 10^{-24} \sqrt{\frac{\Delta E_\odot}{10^{-12}} \frac{1\text{ s}}{\tau}} \left(\frac{1\text{ kpc}}{d}\right)\left(\frac{1\text{ kHz}}{\nu}\right). \tag{26}$$

where $\Delta E_\odot = \Delta E_{GW}/M_\odot c^2$. This quantity is unknown. Therefore to assess the detectability of a

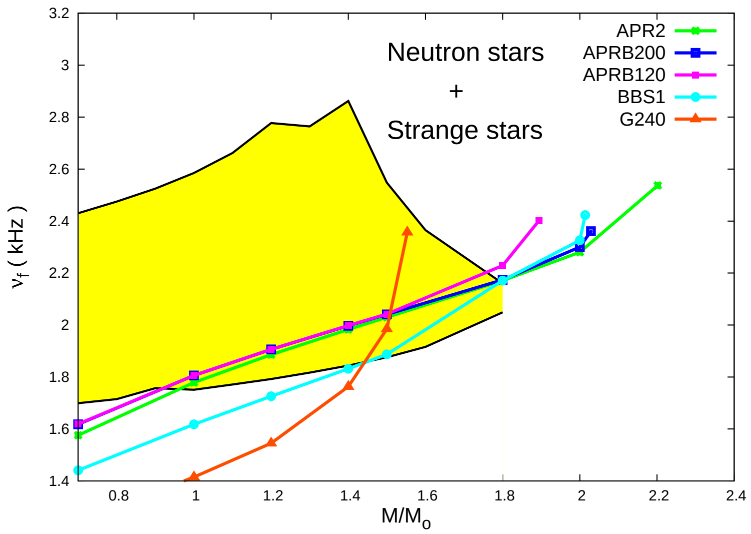

Figure 5. The frequency of the fundamental mode is plotted as a function of the mass of the star, for neutron/hybrid stars (continuous lines) and for strange stars modeled using the MIT bag model, spanning the set of parameters indicated in the range allowed by high energy experiments (dashed region).

signal we can only evaluate how much energy should be emitted in a given mode, in order for the signal to be detected by a given detector with an assigned signal to noise ratio (S/N)

$$\left(\frac{S}{N}\right)^2 = \frac{4Q^2}{1+4Q^2}\frac{\mathcal{A}^2\tau}{2S_n}. \tag{27}$$

In this equation $Q = \pi\nu\tau$ is the quality factor and S_n is the detector spectral noise density. Assuming as a bench-mark for $\Delta E_\odot$ the energy involved in a typical pulsar glitch, in which case a mature neutron star might radiate an energy of the order of $\Delta E_{GW} = 10^{-13}M_\odot c^2$, and assuming $\nu \sim 1500$ Hz, $\tau \sim 0.1$ s, $d = 1$ kpc, we find $\mathcal{A} \approx 5\times10^{-24}$. Such a signal is too weak to be seen by actual detectors, therefore we conclude that 3rd generation detectors are needed to detect signals from old neutron stars. More promising are the oscillations of newly-born neutron stars; indeed, since a NS forms as a consequence of a violent, and generally non-symmetric event – the gravitational collapse – a fraction of its large mechanical energy may go into non-radial oscillations and would be radiated in gravitational waves. Thus, during the first few seconds of the NS life more energy could be stored in the pulsation modes than when the star is cold and old. In addition, during this time the star is less dense than at the end of the evolution; consequently, the frequencies of the modes which depend on the stellar compactness (as for instance the f-mode) are lower and therefore span a frequency range were the detectors are more sensitive (Ferrari, Miniutti & Pons 2003). For instance, if we assume that an energy $\Delta E_{GW} = 1.6\cdot10^{-9}\ M_\odot c^2$ is stored in the f-mode of a neutron star just formed in the Galaxy, the emitted signal would be detectable with a signal to noise ratio $S/N = 8$ by advanced Virgo/LIGO, and with $S/N = 2.7$ by Virgo+/LIGO, the upgraded configurations now in operation.

4. Stellar perturbations and magnetar oscillations

Magnetars are neutron stars whose magnetic field is, according to current models, as large as 10^{15} G (Thompson & Duncan 1993, 2001). During the last three decades some very interesting astrophysical events have been observed which are connected to magnetar activity and stellar pulsations. They involve Soft Gamma Repeaters (SGRs), which are thought to be magnetars; these sources occasionally release bursts of huge amount of energy ($L \simeq 10^{44} - 10^{46}$ ergs/s), and these giant flares are thought of being generated from large-scale rearrangements of the inner field, or catastrophic instabilities in the magnetosphere (Thompson & Duncan 2001; Lyutikov 2003). Up to now, three of these events have been detected: SGR 05026-66 in 1979, SGR 1900+14 in 1998 and SGR 1806-20 in 2004. In two of them (SGR 1900+14 and SGR 1806-20), a tail lasting several hundred seconds has been observed, and a detailed study of this part of the spectrum has revealed the presence of quasi-periodic oscillations (QPOs) with frequencies

$$18,\ 26,\ 30,\ 92,\ 150,\ 625 \quad \text{and} \quad 1840 \quad \text{Hz}$$

for SGR 1806-20 (Watts & Strohmayer 2006), and

$$28,\ 53,\ 84 \quad \text{and} \quad 155 \quad \text{Hz}$$

for SGR 1900+14 (Strohmayer & Watts 2006). The discovery of these oscillations stimulated an interesting and lively debate (still ongoing) among groups working on stellar perturbations, about the physical origin of these sequences. Of course in order to study these oscillations the magnetic field and its dynamics have to be included in the picture, while rotation plays a less important role, since observed magnetars are all very slowly rotating. The problem presents extreme complexity both at conceptual and at computational levels; therefore it is usually approached using simplifying assumptions and/or approximations. Some of the studies try to explain the observed modes in terms of torsional oscillations of the crust (Samuelsson & Andersson 2007; Sotani, Kokkotas & Stergioulas 2007), others attribute the observed spectra to global magneto-elastic oscillations (Glampedakis, Samuelsson & Andersson 2006), still others investigate the interaction between the torsional oscillations of the magnetar crust and a continuum of magnetohydrodynamic modes (the Alfven continuum) in the fluid core (Levin 2007; Sotani, Kokkotas & Stergioulas 2008; Colaiuda, Beyer & Kokkotas 2009; Cerdá-Durán, Stergioulas & Font 2009; Colaiuda & Kokkotas 2010; Gabler et al. 2011) using different approaches and approximations. In particular, in (Colaiuda & Kokkotas 2010) the torsional oscillations of a magnetar have been studied in a general relativistic framework, perturbing Einstein's equations in the Cowling approximation, i.e. neglecting gravitational field perturbations. By this approach the crust-core coupling due to the strong magnetic field has been shown to be able to explain the origin of the observed frequencies, at least for SGR 1806-20, if a suitable stellar model is considered. With this identification, constraints on the mass and radius of the star, and consequently on the EOS in the core, can be set; estimates of the crust thickness and of the value of the magnetic field at the pole can also be inferred.

Thus, the theory of stellar perturbations has been generalized to magnetized stars, although for now only with considerable restriction, since only torsional oscillations have been considered (i.e. axial perturbations) and only in the Cowling approximation. Nevertheless, it already provides very interesting information on the dynamics of these stars and allows us to confront the predictions with astronomical observations.

5. Concluding remarks

I would like to conclude this review by mentioning the fact that the theory of perturbations of rotating stars has not been developed to the same extent as the theory of non-rotating stars. The main reason is that the mathematical tools appropriate for a successful variable separation has not been found yet. When the perturbations of a non-rotating black hole are studied, separation of variables is achieved by expanding all tensors in tensorial spherical harmonics. In the case of Kerr perturbations, namely of perturbations of an axisymmetric, Petrov type D background, the same result is obtained by expanding the Newman-Penrose quantities in oblate spheroidal harmonics. When perturbing a rotating star, i.e. an axisymmetric solution of Einstein+hydro equations, an expansion in terms of tensorial spherical harmonics leads, as we have seen in the case of slow rotation in Section 2.2, to a coupling between polar and axial perturbations. If rotation is not slow, the number of couplings to be considered increases to such an extent that the problem becomes untreatable, both from a theoretical and from a computational point of view. One may argue that, since the background of a rapidly rotating star is not spherically symmetric, tensorial spherical harmonics are unappropriate, and this is certainly true. However, even if we try to use the oblate spheroidal harmonics and the Newmann Penrose formalism we fail: the coupling between the metric and the fluid makes the separation impossible, at least in terms of these harmonics (unlike the Kerr metric, the metric describing a star is not of Petrov type D). For this reason, perturbations of rotating stars have been studied either in the slow rotation regime, or using the Cowling approximation, which neglects spacetime perturbations, or using other simplifying assumptions. For instance, as far as the mode calculation is concerned, the Cowling approximation allows determination with reasonable accuracy the frequency of the higher order p-modes, of the g-modes and of the inertial modes, like the r-modes, thus allowing us to gain information on the onset of related instabilities. Conversely, the determination of the f-mode frequency, which is so important from the point of view of gravitational wave emission, is not very precise, leading to errors as large as $\sim 20\%$.

However, it should be mentioned that non-linear simulations of rotating stars are producing very interesting results; for instance in a recent paper Stergioulas and collaborators (Zink et al. 2010) have been able to follow the frequency of the non-axisymmetric fundamental mode of a sequence of rotating stars with increasing angular velocity, up to the onset of the CFS instability, making also very optimistic estimates of the amount of gravitational radiation which could be emitted in the process. To describe matter in the neutron star they use a simple model (a polytropic equation of state and uniform rotation); however, their result indicates that numerical relativity is making giant steps in this field. Thus, supercomputers are making accessible very complex problems, which only ten years ago one would not have dreamed of solving; however, perturbation theory still remains a very powerful tool to investigate many physical problems and it should be used in parallel with the numerical work to gain a deeper insight into the physics of stellar oscillations.

References

Andersson N., 1994, CQG, 11, 3003
Andersson N., Araujo M.E., Schutz B.F., 1993a, CQG, 10, 735
Andersson N., Araujo M.E., Schutz B.F., 1993b, CQG, 10, 757
Andersson N., Araujo M.E., Schutz B.F., 1993c, Phys. Rev. D, 49, 2703
Andersson N., Kokkotas K.D., 1998, MNRAS, 299, 1059
Andersson N., Thylwe K., 1994, CQG, 11, 2991;
Baiotti L., Hawke I., Rezzolla L., Schnetter E., 2005, Phys. Rev. Lett., 94, 131101
Benhar O., Berti E., Ferrari V., 1999, MNRAS, 310, 797
Benhar O., Ferrari V., Gualtieri L., 2004, Phys. Rev. D70, 124015
Benhar O., Ferrari V., Gualtieri L., Marassi S., 2007, GRG, 39, 1323
Campolattaro A., Thorne K.S., 1970, ApJ, 159, 847
Cerdá-Durán P., Stergioulas N., Font J. A., 2009, MNRAS, 397, 1607
Chandrasekhar S., 1984, The mathematical theory of black holes, Claredon Press, Oxford
Chandrasekhar S., Detweiler S.L., 1975, Proc. R. Soc. Lond., A344, 441
Chandrasekhar S., Ferrari V., 1990a, Proc. R. Soc. Lond., A428, 441
Chandrasekhar S., Ferrari V., 1990b, Proc. R. Soc. Lond., A432, 247
Chandrasekhar S., Ferrari V., 1991a, Proc. R. Soc. Lond., A435, 645
Chandrasekhar S., Ferrari V., 1991b, Proc. R. Soc. Lond., A433, 423
Chandrasekhar S., Ferrari V., 1991c, Proc. R. Soc. Lond., A434, 449
Chandrasekhar S., Ferrari V., 1992, Proc. R. Soc. Lond., A437, 133
Chandrasekhar S., Ferrari V., Winston R., 1991, Proc. R. Soc. Lond., A434, 635
Colaiuda A., Beyer H., Kokkotas K.D., 2009, MNRAS, 396, 1441
Colaiuda A., Kokkotas K.D., 2010, arXiv:1012.3103v2
Cowling T.G., 1942, MNRAS, 101, 367
Cutler C., Lindblom L., 1987, ApJ, 314, 234
Fano U., 1985, Phys. Rev. A, 32, 617
Ferrari V., Miniutti G., Pons J.A., 2003, MNRAS, 342, 629
Ferrari V., Mashhoon B., 1984a, Phys. Rev. D, 30, 295
Ferrari V., Mashhoon B., 1984b, Phys. Rev.Lett, 52, 1361
Gabler M., Cerdá Durán P., Font J.A., Muller E., Stergioulas N., 2011, MNRAS, 410, L37
Glampedakis K., Samuelsson L., Andersson N., 2006, MNRAS, 371, L74
Ipser J.R., Thorne K.S., 1973, ApJ, 181, 181
Landau L.D., Lifshitz E.M., 1975, The classical theory of fields, New York: Pergamon Press
Levin Y., 2007, MNRAS, 377, 159
Lindblom L., Detweiler S.L., 1983, ApJ Suppl., 53, 73
Lyutikov M., 2003, MNRAS, 346, 540
Kokkotas K.D., 1994, MNRAS, 268, 1015
Kokkotas K.D., Schutz B.F., 1992, MNRAS, 255, 119
Pandharipande V.R., 1971a, Nucl. Phys. A, 174, 641
Pandharipande V.R., 1971b, Nucl. Phys. A, 178, 123
Pandharipande V.R., Smith R.A., 1975, Phys. Lett., 59, 15
Press W.H., 1971, ApJ, 170, L105

Pudliner B.S., Pandharipande V.R., Carlson J., Pieper S.C., Wiringa R.B., 1995, Phys. Rev. C, 56, 1720
Regge T., Wheeler J.A., 1957, Phys. Rev., 108, 1063
Samuelsson L., Andersson N., 2007, MNRAS, 374, 256
Schutz B.F., Will C.M., 1985, ApJ, 291, L33-L36
Sorkin R., 1991, Proc. R. Soc. Lond., A435, 635
Sotani H., Kokkotas K.D., Stergioulas N., 2007, MNRAS, 375, 261
Sotani H., Kokkotas K.D., Stergioulas N., 2008, MNRAS, 385,L5
Strohmayer T.E., Watts A.L., 2006, ApJ 653, 593
Teukolsky S., 1972, Phys. Rev. Lett., 29, 1114
Teukolsky S., 1973, ApJ, 185, 635
Thompson C., Duncan R.C., 1993, ApJ, 408, 194
Thompson C., Duncan R.C., 2001, ApJ, 561, 980
Thorne K.S., 1969a, ApJ, 158, 1
Thorne K.S., 1969b, ApJ, 158, 997
Thorne K.S., Campolattaro A., 1967, ApJ, 149, 591
Thorne K.S., Campolattaro A., 1968, ApJ, 152, 673
Vishveshwara C.V., 1970, Phys. Rev.D, 1, 2870
Walecka J.D., 1974, Ann. Phys., 83, 491
Watts A.L, Strohmayer T.E., 2006, ApJ, 637, L117
Wiringa R.B., Fiks V., Fabrocini A., 1988, Phys. Rev. C, 32, 1057
Wiringa R.B., Stoks V.G.J., Schiavilla R., 1995, Phys. Rev. C, 51, 38
Zerilli J.F., 1970a, Phys. Rev. D, 2, 2141;
Zerilli J.F., 1970b, Phys. Rev. Lett., 24, 737
Zink B., Korobkin O., Schnetter E., Stergioulas N., 2010, Phys. Rev. D81, 084055