



## Problems of collisional stellar dynamics

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**Abstract.** The discovery of dynamical friction was Chandrasekhar's best known contribution to the theory of stellar dynamics, but his work ranged from the few-body problem to the limit of large  $N$  (in effect, galaxies). Much of this work was summarised in the text "Principles of Stellar Dynamics" (Chandrasekhar 1942, 1960), which ranges from a precise calculation of the time of relaxation, through a long analysis of galaxy models, to the behaviour of star clusters in tidal fields. The later edition also includes the work on dynamical friction and related issues. In this review we focus on progress in the collisional aspects of these problems, i.e. those where few-body interactions play a dominant role, and so we omit further discussion of galaxy dynamics.<sup>2</sup> But we try to link Chandrasekhar's fundamental discoveries in collisional problems with the progress that has been made in the 50 years since the publication of the enlarged edition.

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### 1. Introduction

Chandrasekhar's "Principles of Stellar Dynamics" is not his best-known text, but it had few competitors for many years, and covered a broad range of topics. The later edition (Chandrasekhar 1960) added a number of lengthy and significant papers mainly on the statistical approach to collisional stellar dynamics, and was published just over 50 years ago. In this review we consider

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<sup>2</sup>There is one other such problem to which Chandrasekhar contributed, though the paper in question (Chandrasekhar 1944) was not reprinted in the book. See Section 2. For more on the collisionless problems studied by Chandrasekhar, see the paper by N. Wyn Evans (2011) in the present volume.

a few of the topics considered by Chandrasekhar, and try to connect his view of the subject with current research.

Since the book is not so well known, and has been virtually supplanted by the book by Binney and Tremaine (Binney & Tremaine 1987, 2008), it is worth looking over its contents list. After an observational review, the theory begins with a careful derivation of the relaxation time of stellar systems, including all the geometry of two-body encounters and a discussion of the origin of the Coulomb logarithm. It contains a derivation of a formula for what came to be known as “dynamical friction”, possibly Chandrasekhar’s most significant and enduring discovery in the field of stellar dynamics. The history and context of Chandrasekhar’s work in these two topics is nicely discussed in Padmanabhan (1996), and in the present paper we consider more recent developments in the theory of dynamical friction in Section 4.

In Chandrasekhar’s book there follows two chapters on collisionless stellar dynamics, or rather the dynamics of galaxies. After some preliminaries, the first of these considers the problem of determining what galactic potentials are consistent with the assumption of a Schwarzschild (Gaussian) distribution of velocities, while the second turns to the problem of spiral structure. Then there is a long and interesting chapter on collisional stellar dynamics; specifically, the dynamics of star clusters, a subject which we review here in Section 3.

Apart from two appendices, at this point the old and new editions of the book differ. The latter now includes several reprints, some on dynamical friction (a topic we take up here in Section 4), and a final long paper on the statistics of the gravitational field of a distribution of point masses, together with its applications to dynamical friction and star clusters. It is amusing to find that Chandrasekhar titled this last paper “New Methods in Stellar Dynamics”. One wonders if this was a conscious echo of Poincaré’s famous “Les Méthodes Nouvelles de la Mécanique Celeste” (see also Hénon 1967). At any rate, one of Chandrasekhar’s applications of his theory is the starting point for the next section of this review.

## 2. The Dynamics of Binary Stars

We begin with a slim paper “On the Stability of Binary Systems” (Chandrasekhar 1944). It did not make it into his book, but it appears to be Chandrasekhar’s only work on a topic which has become one of the pillars of collisional stellar dynamics. In Chandrasekhar’s paper, it is set in the context of a critique by Ambartsumian (Ambartsumian 1937) of an older idea by Jeans, who had used information on the distribution of binary stars to argue that the Milky Way was well relaxed.

### 2.1 The statistical effect of encounters

Chandrasekhar’s estimate for the disruption time scale,  $\tau$ , of a binary was based on his theory of the two-point distribution for the gravitational acceleration due to a random distribution of stars,

which led to the formula

$$\tau = \frac{(M_1 + M_2)^{1/2}}{4\pi G^{1/2} M N a^{3/2}},$$

where  $M_1, M_2$  are the component masses,  $a$  is the semi-major axis, and  $M, N$  are the average individual mass and number density of the field stars. Ambartsumian's formula, by contrast, was

$$\tau = \frac{v}{4\pi G M a N \ln \left( 1 + \frac{a^2 v^4}{4G^2 M^2} \right)},$$

where  $v$  is some average speed which we take here to be the velocity dispersion. Notice that, by contrast, Chandrasekhar's formula includes no information on the velocity dispersion, because the underlying theory describes the spatial correlation of fluctuations but not their temporal correlation. He gives the impression that the absence of any dependence on the velocities is a merit, but it later turned out that the velocities of the stars are crucial. Jeans himself (Jeans 1918) had argued (incorrectly, as it later emerged) that encounters with field stars would lead to equipartition of kinetic energies, giving all binaries a period of order

$$P = \frac{G(M_1 + M_2)}{v^3}. \quad (1)$$

This conclusion was, however, turned upside down by Gurevich & Levin (1950), who used arguments akin to those of Ambartsumian and obtained formulae for the average rate of change of the binding energy of a binary as a result of encounters. They concluded that, if a binary has period much longer than equation 1, then its period will tend to become longer still (eventually leading to disruption), while if its period is much shorter then it becomes still shorter. Their conclusion was correct, and was arrived at independently by Hills (Aarseth & Hills 1972; Hills 1975) using numerical methods and by Heggie (Heggie 1975) using mainly analytical approximations. Heggie also used the terms "hard" and "soft" to signify binaries whose internal binding energy,  $\varepsilon$ , was larger or smaller, respectively, than the mean kinetic energy of the field stars.

The case of equal masses has been worked out in a lot of detail, especially in a series of papers by Hut and colleagues, much of it summarised with tables, figures and formulae in Heggie & Hut (1993). Applications, of course, require unequal masses in general, and here our knowledge is much more patchy. Heggie, Hut & McMillan (1996) were able to give a general formula for exchange encounters with hard binaries (i.e. encounters in which the incoming third star takes the place of one of the original binary components). They used analytical arguments to establish the mass dependence for extreme cases (e.g. one component of very low mass), and filled in the gaps by interpolation in results of numerical experiments. For this purpose they used the starlab package (<http://www.manybody.org/manybody/starlab.html>, McMillan & Hut (1996)), which has very well organised tools for the computation of scattering cross sections. Large numbers of other results for various specific combinations of masses will be found scattered in the literature, including Sigurdsson & Phinney (1993) and especially the compendious book of Valtonen & Karttunen (2006).

A number of extreme parameter ranges have become important for astrophysical reasons, and also these are situations in which the complexities of the mass-dependence of a cross section may be considerably reduced. On the other hand, as we shall see, the situation can be considerably richer than the simple notion that soft binaries soften and hard binaries harden.

The case of a third body (intruder) of relatively low mass has been studied in the context of the hardening of a black hole-black hole binary in a galactic nucleus (Mikkola & Valtonen 1992). It led to an interesting debate (Hills 1990; Gould 1991) on whether it is really true that hard binaries (defined by the ratio of the binding energy of the binary to the energy of relative motion of the intruder) tend to harden and soft binaries tend to soften. Hills had argued that it was the ratio of *speeds* that mattered, but Quinlan (1996) eventually vindicated the earlier position. There is now a considerable literature on the problem of a massive binary in a system of particles of low mass which uses  $N$ -body simulations rather than cross sections.

Another important context where the distinction between fast encounters and energetic encounters is significant is the study of stellar encounters with planetary systems. This has been studied by several groups, including Laughlin & Adams (1998); Hills & Dissly (1989); Malmberg, Davies & Heggie (2010); Spurzem et al. (2009), but here we focus briefly on the study of Fregeau, Chatterjee & Rasio (2006). In the case under consideration, let us denote the incomer velocity which distinguishes hard from soft binaries (in the sense of energies) by  $v_c$  (the “critical” velocity, in the sense that slower encounters cannot destroy the binary). Because the planetary mass is so low,  $v_c$  is much smaller than the orbital speed of the planet ( $v_{orb}$ ). These authors find that, indeed, when the encounter speed is less than a speed of order  $v_c$ , the average change in the binding energy of the binary is positive, i.e. the encounter hardens the binary. But at the same time the planetary system has been destroyed, because the most likely outcome of a close encounter in this regime is an exchange encounter leaving the two stars bound. Similarly, in the regime of encounter speeds between  $v_c$  and  $v_{orb}$  an encounter indeed tends to soften a planetary system, but not to disrupt it (until encounter speeds of order  $v_{orb}$  are reached). A careful reading of Fregeau, Chatterjee & Rasio (2006) is recommended for a proper appreciation of the issues.

The last case of extreme masses that we shall mention is another highly topical one: that of a single black hole encountering a binary with stellar-mass components. This is thought to be of importance in the creation of high-velocity stars by scattering off the black hole at the Galactic Centre (Hills 1988). The literature is considerable, but among those studies focusing on the three-body aspects of the problem are Zhang, Lu & Yu (2010); Gvaramadze, Gualandris & Portegies Zwart (2009); Sari, Kobayashi & Rossi (2010); Gualandris, Portegies Zwart & Sipior (2005); Yu & Tremaine (2003).

## 2.2 Wide binaries in the Milky Way

Chandrasekhar’s interest in binary stars was focused on the dynamical evolution of field binaries, a topic which remains lively up to the present day, with an extensive literature, especially on the observational side. Before turning to recent developments, however, it is worth mentioning that

the topic had already been considered, before the work of Chandrasekhar and Ambartsumian, by Öpik (1932).

This thread of research was then taken up by Oort (1950) who, like Öpik, was concerned with binaries with one massless component (a comet or meteoroid). While all these studies used analytical estimates, soon after this numerical integrations came into routine use, and were applied to this problem by Yabushita (1966) and Cruz-González & Poveda (1971). The latter authors found that the lifetime exceeded the estimates given by any of the previous theories which they tested. There were also theoretical developments, however. Except for Chandrasekhar's theory, that earlier work had been based on the computation of the mean square change of velocity (i.e. the relative velocity between the two components of a binary), or the mean change in the energy of the binary. As Chandrasekhar himself would have recognised, however, it is also necessary to take into account the second moment of the change in energy (i.e. its mean square value), to construct a kinetic theory based on a Fokker-Planck treatment. This was accomplished in King (1977) and Retterer & King (1982).

Further theoretical developments have mainly involved improvements in the physical model, i.e. the inclusion of significant additional processes, such as encounters with giant molecular clouds (Weinberg, Shapiro & Wasserman 1987) and with dark matter particles (Wasserman & Weinberg 1987). Nor are the wide binaries of the Galactic field lacking interest even after they have dissolved. Then they are also strongly subject to galactic perturbations, which impose an interesting (and potentially detectable) correlation in density with a peak when the two components have separated by 100-300pc (Jiang & Tremaine 2010). Finally, it is not self-evident how wide binaries can emerge from the relatively dense star-cluster environment in which most stars are thought to form, and indeed it seems likely that significant numbers form during the cluster dissolution process itself (Kouwenhoven et al. 2010).

### 3. The dynamics of star clusters

The title of this section is also the title Chandrasekhar chose for the last chapter of his book. As usual, it opens with a number of generalities, but then it moves on to the important problem of the escape rate from a star cluster, including the differential escape of stars of low and high mass. Implicit in this theory is the assumption that the cluster is isolated, but the next section of his book moves on to consider the effect on a cluster of its galactic environment. This section begins with an excellent derivation of equations of motion, "energy"-integral and virial theorem.

#### 3.1 Tidal stability

After the preliminaries, Chandrasekhar takes up the stability of star clusters, using as his model an ellipsoidal cluster of uniform density  $\rho$ . The reason for this is that the gravitational field (including the tidal field of the galaxy) becomes linear, and the motions of the stars can be computed explicitly. The frequencies become imaginary when  $\rho$  is sufficiently low, and Chandrasekhar in-

interprets this as the onset of instability. A somewhat similar approach was taken by Angeletti, Capuzzo-Dolcetta & Giannone (1983), who studied orbits in the nearly constant-density core of a cluster. They used Floquet analysis to determine the stability limit and were thus able to extend results to the case of a cluster on a non-circular galactic orbit.

This section of Chandrasekhar's book is of particular interest to the author of the present paper because it turns out to be possible to construct a self-consistent ellipsoidal model of uniform density by superposition of these exact orbits (Mitchell & Heggie 2007), though they are indeed unstable when the density is low enough (Fellhauer & Heggie 2005). Though these models are artificial<sup>3</sup>, they are of interest because examples of self-consistent cluster models in a tidal field are rare (Heggie & Ramamani 1995; Bertin & Varri 2008; Varri & Bertin 2009). These models also give a pointer for the construction of better models of star clusters than any in existence, in the following sense. These models consist of the familiar galactic epicycles, but modified by the attraction of the cluster. They are therefore retrograde orbits, and it has been known for a long time that there should exist *stable* retrograde orbits in the vicinity of a star cluster, but outside its tidal radius (Hénon 1970). Thus one can imagine a sequence of self-consistent cluster models with varying proportions of stars inside and outside the cluster tidal radius, with (say) the models of Heggie & Ramamani (1995) (which are generalised King models) at one end of the sequence, and the models of Mitchell & Heggie (2007) at the other.

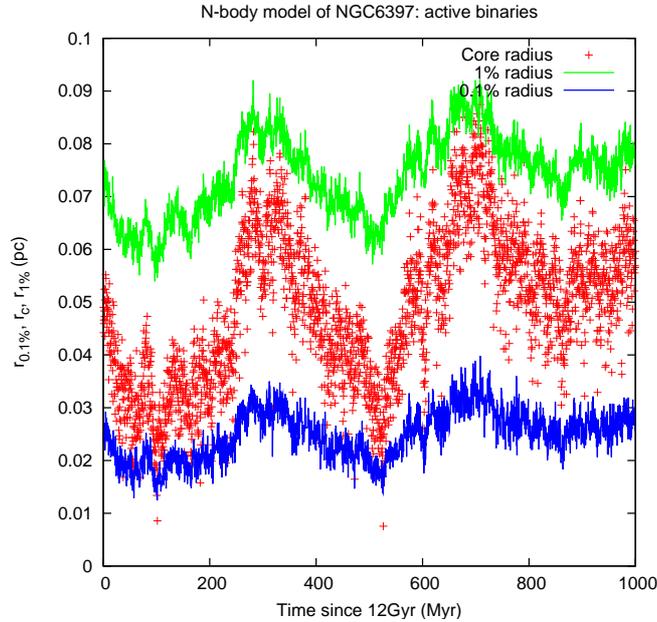
### 3.2 Fokker-Planck dynamics

Towards the end of his chapter on star clusters, in which Chandrasekhar has discussed both escape and relaxation, he laments, "A rigorous theory of galactic clusters must therefore take both these factors into account. But such a theory is not yet available." It was not too long in coming, the essential formalism being established by Kuzmin (1957). But its power was first demonstrated by Hénon (1961), in a landmark paper which, among other things, produced a solution of the Fokker-Planck equation (for the relaxation) with a tidal boundary condition (producing escape). Not only this, but Hénon also realised the critical role played by binaries.

Hénon's model was of a very special type, but one which all reasonable solutions would approach asymptotically. It took almost another 20 years before a general numerical solution of the equation became feasible (Cohn 1979), though initially restricted to the case of stars of equal mass, as in Hénon's model. This refinement was added relatively quickly, however (Merritt 1981, 1983), albeit in the context of *galaxy* clusters (as opposed to galactic ones). The subsequent development of this tool was steady: rotation (Goodman 1983), binary stars formed in three-body encounters (quoted in Cohn 1985) or those formed tidally (Statler, Ostriker & Cohn 1987) or primordially (Gao et al. 1991) and stellar evolution (Kim, Chun & Min 1991) were all added;

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<sup>3</sup>The later paper has never been cited so far, much as the referee predicted. The present paper will probably provide its one and only citation.



**Figure 1.** Evolution of the core in a direct  $N$ -body model of the globular cluster NGC6397 (from Heggie & Giersz (2009)). From top to bottom, the plotted radii are the 1% Lagrangian radius (i.e. the radius of a sphere, centred at the densest part of the cluster, which contains 1% of the total mass), the core radius (i.e. the radius at which the density of the cluster falls to a certain fraction of its central value, though it is actually calculated by a different procedure), and the 0.1% Lagrangian radius. Radii are given in parsecs, and the horizontal axis is time (in units of 1Myr) after the present. The cluster core is alternately compact and more extended, on a time scale which is long by comparison with the central relaxation time; this, and other arguments, suggest that the oscillations are essentially gravothermal.

until it became a tool which could be applied to the modelling of individual star clusters and quite detailed comparison with observations.

This was not the first kind of code to reach this goal, however. In advance of the development of Fokker-Planck methods was a method of treating the dynamics of a star cluster as if it was a self-gravitating gas (Larson 1970). This technique developed with comparable rapidity, and after only 10 years it was possible to produce synthetic surface density profiles for comparison with observation (Angeletti, Dolcetta & Giannone 1980). These gas models remained of importance, and were responsible for the discovery of gravothermal oscillations (Sugimoto & Bettwieser 1983), which are the response of a system to an unstable balance between the relaxation-driven flow of energy and the redistribution of energy by binary interactions in the core. Interest in this behaviour slowly waned, but has recently been invoked as a significant mechanism for understanding the variety of surface brightness profiles exhibited by well observed star clusters (Fig. 1).

### 3.3 Monte Carlo models

From the numerical point of view, both the Fokker-Planck and gas models are of finite difference type. It is also possible to solve the former equation with at least two kinds of Monte Carlo technique. One of these was pioneered by Spitzer and his students (see Spitzer & Hart (1971) and subsequent papers in the series). Its last serious application appeared many years ago (Spitzer & Mathieu 1980), and it is probably ripe for revival, as it better adapted to some important situations (e.g. a time-dependent tide) than some competitors.

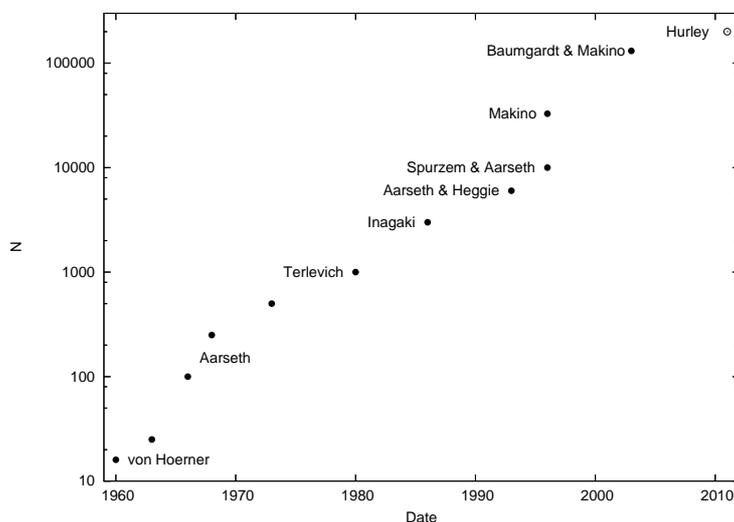
An alternative Monte Carlo technique was developed at about the same time (Hénon 1967, 1971), but has been taken up and developed by others, right up to the present (Stodolkiewicz 1982; Giersz 2006; Chatterjee et al. 2010). It now includes a rich mix of important ingredients, not only relaxation and escape, but also the internal (stellar) evolution of single stars and binaries, as well as interactions involving binaries. Despite its limitations to a steady tidal field, spherical symmetry and zero rotation, it is the method of choice for studying virtually all globular star clusters, because it is so fast, without sacrificing much realism. Even the evolution of a rich star cluster like 47 Tuc, which is thought to have almost  $2 \times 10^6$  stars and a few percent of binaries, can be modelled for a Hubble time in less than a week on an ordinary computer (Giersz & Heggie 2010). Such modelling makes possible a range of investigations at the interface with observations, and is very useful for the planning and interpretation of some kinds of observational programmes, such as searches for radial velocity binaries (Sommariva et al. 2009). The speed is important, because we do not know *ab initio* what initial conditions to use to match a given cluster. Repeated trial and error, or grid searches, require a fast method.

### 3.4 *N*-body methods

Naturally enough, there is nothing in *Principles of Stellar Dynamics* which prepares us for the explosion of interest in *N*-body methods in the subject, even if we restrict ourselves to direct summation methods. It started about 20 years after the publication of the book (von Hoerner 1960), and in 50 years has brought us to the point where it begins to be possible to study the entire life history of the easiest globular clusters (Hasani Zonoozi et al. 2011), though most still lie beyond our capabilities.

The main problem is the number of stars,  $N$ . Fig. 2 shows the steady but slow progress that has been made since 1960. The mean mass of the Galactic globular clusters being (Mandushev, Staneva & Spasova 1991) of order  $1.9 \times 10^5 M_\odot$  (and the median mass is lower still), it might be thought that a large fraction of them are within reach of *N*-body simulation. However, they lose large amounts of mass through evolution over about 12Gyr, and so, except for a few sparse and large clusters, the initial stages of the evolution prevents them from being simulated in a reasonable time.

Actually, it is not hard to evolve larger models than those shown in Fig. 2 to well beyond



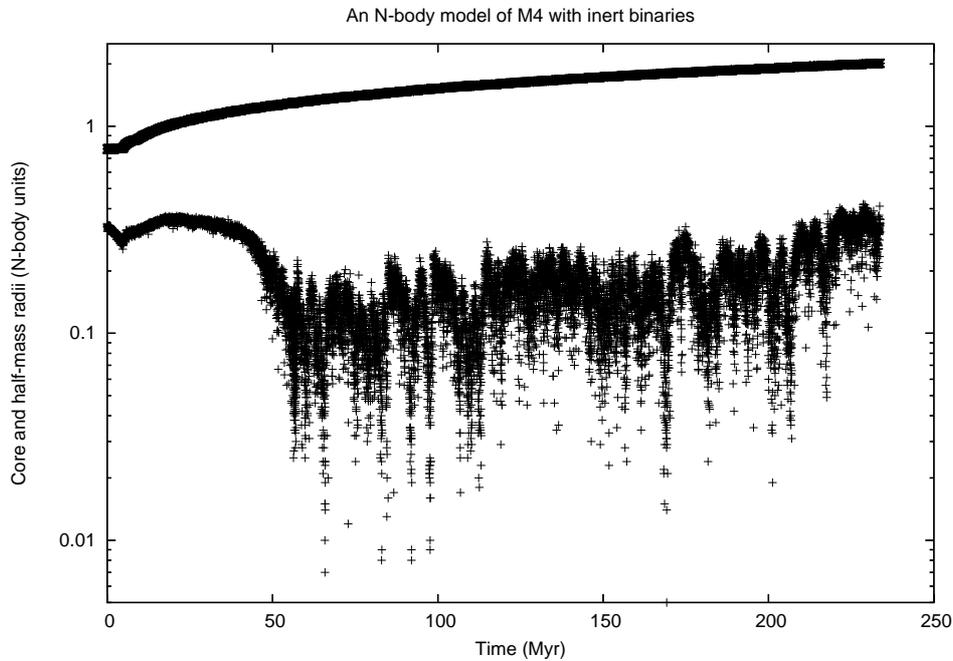
**Figure 2.** Largest direct  $N$ -body simulation to date, plotted against publication date. Only dynamically well evolved simulations (roughly speaking, to or beyond “core collapse”; see, for example, Heggie & Hut (2003)) are included. The last simulation is not yet published at the time of writing (February 2011), in fact see Hurley et al. (2008).

core collapse. Fig. 3 shows a simulation using as initial conditions those suggested for the cluster M4 by Heggie & Giersz (2008), except that there are *no primordial binaries*. If these had been included (and the suggested abundance is only about 7%) the progress of the simulation would have been slower by about a factor of 20.

### 3.5 The escape rate from star clusters

Chandrasekhar (1943a,b) produced two papers on this topic in quick succession. His motivation for this was not simply to understand the lifetime of star clusters, but to elucidate the role played by dynamical friction (Section 4). Dynamical friction is an aspect of two-body relaxation which tends to reduce the energy of stars, especially those of high speed, and which therefore particularly affects escaping stars. Without it, Chandrasekhar showed, the lifetime of a star cluster would be too short to explain the existence of star clusters with ages of order a Gyr. Somewhat analogously, it has also been invoked in studies of the escape of black holes from a galaxy (Kapoor 1985a,b).

In Chandrasekhar’s papers he was, in effect, solving the Fokker-Planck equation assuming that escape took place at some fixed speed (which he estimated from the virial theorem). Similar calculations have been carried out by Spitzer & Harm (1958); King (1960) and Lemou & Chavanis (2010). Long ago, however, King (1958) pointed out a number of shortcomings of the



**Figure 3.** Core- and half-mass radii of an  $N$ -body model with 453 000 particles initially. The initial conditions are described in the text. The model includes stellar evolution and the effect of the Galactic tide, but *no binaries* (initially). The initial brief reduction of the core radius is caused by mass segregation. After about 5 Myr, stellar evolution temporarily halts the collapse of the core, but this resumes, and reaches completion after about 50 Myr. Thereafter the small core is sustained by the formation and dynamical evolution of binary stars, but the core is now unstable to “gravothermal oscillations”. Both stellar evolution (or rather its associated mass loss) and the evolution of binaries increase the energy of the cluster, which leads to the expansion of the half-mass radius. The tidal radius (not shown), which marks the effective boundary between motions dominated by the cluster and those dominated by the Galaxy, decreases by only about 15% in the time shown. The units of radius are “ $N$ -body units” (see Heggie & Hut (2003)). This simulation took about 2 months.

Chandrasekhar model, and proceeded to investigate some of them in subsequent papers. One of these was the spatial inhomogeneity of the star cluster model, which, in the same paper, he investigated by integrating the escape rate over the cluster. The escape rate formula which he integrated was a more primitive estimate than Chandrasekhar’s, dating back to the earlier work of Ambartsumian (1938) and Spitzer (1940). A similar treatment, but based on more elaborate formulae for the local escape rate were later presented by Saito (1976) and Johnstone (1993).

All the formulae in the papers cited are based on the theory of relaxation, and therefore include as a factor the Coulomb logarithm. A completely different view of the situation was taken by Hénon (1960), who showed that relaxation cannot lead to escape from an isolated cluster,

essentially because the “period” of a star’s orbit tends to infinity as its energy approaches zero (from below). He obtained a formula for the escape rate due to discrete, individual encounters (rather than the diffusive effective of many encounters). Tellingly, it does not contain the Coulomb logarithm.

One of the comments made by King (1958) was that, however one computes the escape rate, it will change as the cluster evolves. Spitzer & Shapiro (1972) pointed out that relaxation changes the distribution function of the stars in a cluster, and then a single encounter, as envisaged by Hénon, may raise the energy of a star above the energy of escape. Therefore it is possible that the relaxation time scale does, after all, control the escape rate from an isolated system, as is commonly assumed.

There have been many numerical studies addressing the problem, but we shall mention only one (Baumgardt, Hut & Heggie 2002), which showed that another, additional mechanism is at play. While it is true that most escapers emerge from encounters deep inside the cluster, some occur because this process causes the potential well of the cluster to become shallower, and this in turn causes stars with energies just below the energy of escape to drift across the escape boundary.

All these issues change when one considers a cluster limited by the tidal field of a galaxy. At the simplest level, the energy of escape drops and the rate of escape increases, as was found numerically many years ago (Wielen 1968; Hayli 1970). But the very notion of escape is complicated. It is possible, at least on the standard model of a star cluster on a circular galactic orbit, for an escaper to recede arbitrarily far from the cluster and still return to it (Ross, Mennim & Heggie 1997). Stars can exist on stable orbits with energies above the escape energy, and even on orbits which lie outside the conventional tidal radius of the cluster (Hénon 1970); see also Section 3.1. Such stars have important effects on the observable velocity dispersion profile of a globular cluster (Küpper et al. 2010). Matters are complicated further for clusters on *non-circular* galactic orbits, where there is no conserved quantity analogous to energy (and therefore no notion of escape energy or even of an escape boundary). Nevertheless the common view is that, even in these cases, the time scale of escape is determined by the time of relaxation.

Among early indications that this is not so were numerical results by Vesperini & Heggie (1997), who showed that the escape rate depended systematically on  $N$ , even when scaled by the relaxation time.  $N$ -body models by T. Fukushige and J. Makino (Heggie et al. 1998) showed clearly that escape scales with  $N$  in a different way from relaxation. Further  $N$ -body results (Heggie 2001a,b) gave an escape time scale proportional (empirically) to about  $N^{0.63}$ , whereas in the same units the relaxation time scales approximately as  $N/\log N$ .

The problem was greatly clarified by the work of Baumgardt (2001). He showed that the scaling could be understood by noting that stars may remain inside a static cluster for an arbitrarily long time, even with energies above the escape energy (Fukushige & Heggie 2000), and that this changes the escape time scale from the relaxation time,  $t_r$ , to approximately  $t_r^{3/4} t_{cr}^{1/4}$ , where  $t_{cr}$  is the crossing time. For the range of  $N$  studied in  $N$ -body simulations of the time, this results in a dependence close to  $N^{0.63}$ , in units such that  $t_{cr}$  is constant.

It is the interaction between this buffer of “potential escapers” and the processes of relaxation and escape which complicates the overall escape time scale. The effect of this buffer is sometimes referred to as a “retardation effect”, after a study by King (1959), which was in turn suggested by a remark of Chandrasekhar (1960, p.209). The point is that a star which has gained enough energy to escape may, on its way out of the cluster, experience another encounter which brings it below the escape energy once more. But the  $N$ -dependence of this effect is different from that described in Baumgardt (2001), and there it is not encounters which retain an escaper, but the dynamics of stars in the field of tidal and inertial forces.

The scaling does depend on factors such as the initial concentration of the cluster (Tanikawa & Fukushige 2005, 2010), the extent to which the cluster initially underfills its tidal radius (Gieles & Baumgardt 2008), and the galactic environment (i.e. the strength of the tidal field, Lamers, Gieles & Portegies Zwart (2005)). Amazingly, it does not depend significantly on the assumption that the cluster orbit is circular (Baumgardt & Makino 2003); if the orbit is non-circular, the cluster appears to behave like one on a circular orbit of intermediate radius. Understanding this fact from a theoretical point of view is a significant unsolved problem in this area, despite some empirical advances with the aid of  $N$ -body simulations (Küpper et al. 2010). There is also growing observational evidence on the mass-dependence of cluster disruption, and it is consistent with these theoretical developments (Boutloukos & Lamers 2003; Gieles et al. 2005; Lamers & Gieles 2006), though it has to be recognised that other processes come into play beyond the relatively gentle evaporation of escapers created in encounters. That is a long and old story which we shall not review here.

Equally old and long is the theory of what is called *preferential* or *differential* escape, i.e. its dependence on the mass of the escaper. The common opinion is that the escape rate increases with decreasing mass, but Chandrasekhar’s finding (Chandrasekhar 1960) (p.209f) was more subtle. His result was that the escape rate is fastest for stars with a mass of about 40% of the mean mass. This result was based on the assumption that stars are in energy equipartition in the cluster, which is inconsistent with a fixed escape velocity. Nevertheless the result still turns out to hold in star clusters which include stellar evolution and a low remnant retention fraction (Kruijssen 2009), if the total disruption time of the cluster is short enough.

#### 4. Dynamical friction

This is another subject with a long and rich history, and it takes us beyond star cluster dynamics into the dynamics of galaxies and galaxy clusters. In that context, which we come to at the end of this section, it also takes us away from the collisional problems to which this review has been devoted. Within collisional stellar dynamics, dynamical friction is simply part of the mainstream of the theory of relaxation, and does not often receive separate, explicit mention. There is an interesting experimental check of what is, in effect, the coefficient of dynamical friction in Theuns (1996). Within limits the comparison is satisfactory, but this study also shows that direct comparison is not an easy task.

One current problem of collisional stellar dynamics involving dynamical friction explicitly (and linking with the topics of section 2) is the fate of black holes in merging galaxies. Their evolution was outlined in a famous paper of Begelman, Blandford & Rees (1980), and much subsequent attention has been paid to a protracted period of evolution under the action of dynamical friction, often referred to as “the final parsec problem”. Some theoretical studies relevant to this problem (scattering of low-mass objects off a massive binary) are referred to in Section 2.1, and others which refer specifically to the galactic context include Polnarev & Rees (1994) and Vecchio, Colpi & Polnarev (1994). (Of course it is a big assumption to suppose that this process of the evolution of pairs of black holes can be understood entirely in terms of stellar dynamics; the effect of galactic gas and accretion disks around the black holes may be decisive; but we shall continue to ignore these in our further review.)

In the stellar dynamical problem Chandrasekhar’s formula has been extended in several ways, e.g. to a non-uniform background medium (Just & Peñarrubia 2005), one with an anisotropic velocity distribution (Ideta 2002), or one with a mass spectrum (Ciotti 2010).  $N$ -body techniques are possible, but demanding, because it is known on theoretical grounds that it is necessary to include the effect of the black holes on the stellar distribution self-consistently (Iwasawa et al. 2011; Sesana 2010). To reach a regime which can be scaled robustly to galactic nuclei is a computational challenge comparable to the simulation of globular clusters (Section 3.4). Progress has been faster than for the globular cluster problem, however, partly because there is no need (it is assumed) to follow also a binary population in the stellar distribution (Berczik, Merritt & Spurzem 2005; Berczik et al. 2006; Berentzen et al. 2009).

Black holes are point masses, but the notion of dynamical friction has been extended (in numerous studies) to problems of the orbital evolution of a satellite galaxy within a larger galaxy or halo, i.e. to extended bodies. From the theoretical point of view it seems clear that the behaviour of a rigid satellite (which is the basis of some theoretical studies) may differ essentially from that of a responsive satellite (Fujii, Funato & Makino 2006). A common approach is to use a more-or-less self-consistent  $N$ -body simulation and to summarise the results by a calibration of the Coulomb logarithm in the Chandrasekhar formula; examples include Chan, Mamon & Gerbal (1997); Cora, Muzzio & Vergne (1997); Spinnato, Fellhauer, & Portegies Zwart (2003) and Just et al. (2010).

While dynamical friction, as introduced by Chandrasekhar, is a mechanism of collisional stellar systems, galaxies are collisionless (at least, in the regime under discussion here). Indeed, since Chandrasekhar’s time, it has become clear that there is a *collective* process which governs such phenomena as the decay of the orbit of a satellite galaxy in the halo of its parent galaxy (Tremaine & Weinberg 1984; Weinberg 1986; Colpi & Pallavicini 1998). It might even be concluded that the physical phenomenon which causes the orbital decay of satellite galaxies has no deeper connection with dynamical friction than the same dependence on the basic scales of density and velocity dispersion (which allows it to be expressed by a suitable choice of the Coulomb logarithm). Even more tenuous is the link between Chandrasekhar’s theory and the decay of a satellite in a partly or purely *gaseous* medium, though this too is often referred to as “dynamical friction”. While there is some danger of confusing the underlying physics, perhaps it is a measure

of the appeal of Chandrasekhar's discovery that the term "dynamical friction" has been extended to encompass such a diversity of astrophysical processes.

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