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Stability of relativistic stars

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Abstract. Stable relativistic stars form a two-parameter family, parametrized by mass and angular velocity. Limits on each of these quantities are associated with relativistic instabilities discovered by Chandrasekhar: A radial instability, to gravitational collapse or explosion, marks the upper and lower limits on their mass; and an instability driven by gravitational waves may set an upper limit on their spin. Our summary of relativistic stability theory given here is based on and includes excerpts from the book *Rotating Relativistic Stars*, by the present authors (Friedman & Sterigioulas 2011).

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1. Introduction

A neutron star in equilibrium is accurately approximated by a stationary self-gravitating perfect fluid.[1] The character of its oscillations and their stability, however, depend on bulk and shear viscosity, on the superfluid nature of its interior, and – for modes near the surface – on the properties of the crust and the strength of its magnetic field.

The stability of a rotating star is governed by the sign of the energy of its perturbations; and the amplitude of an oscillation that is damped or driven by gravitational radiation is governed by the rate at which its energy and angular momentum are radiated. Noether's theorem relates the stationarity and axisymmetry of the equilibrium star to conserved currents constructed from the

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[1] Departures from the local isotropy of a perfect fluid are associated with the crust; with magnetic fields that are thought to be confined to flux tubes in the superfluid interior; and with a velocity field whose vorticity is similarly confined to vortex tubes. Departures from perfect fluid equilibrium due to a solid crust are expected to be smaller than one part in $\sim 10^{-3}$, corresponding to the maximum strain that an electromagnetic lattice can support. The vortex tubes are closely spaced; but the velocity field averaged over meter scales is that of a uniformly rotating configuration. Finally, the magnetic field contributes negligibly to the pressure support of the star, even in magnetars with fields of 10^{15} G.

perturbed metric and fluid variables. Their integrals, the canonical energy and angular momentum on hypersurface can each be written as a functional quadratic in the perturbation, and the conservation laws express their change in terms of the flux of gravitational waves radiated to null infinity.

We begin with an action for perturbations of a rotating star from which these conserved quantities are obtained. The action was introduced by Chandrasekhar and his students in the Newtonian approximation (Chandrasekhar 1964; Clement 1964; Lymden Bell & Ostriker 1967), and its generalization to the exact theory was initially due to Chandra, in his pioneering paper on the stability of spherical relativistic stars (Chandrasekhar 1964). Several authors, including Taub, Carter, Chandrasekhar, Friedman, and Schutz (Taub 1954, 1969; Carter 1973; Chandrasekhar & Friedman 1972 a & b, Friedman & Schultz 1975; Friedman 1978) extended it to a Lagrangian formalism for rotating stars in general relativity.

We next review local stability to convection and to differential rotation. A spherical star that is stable against convection is stable to all nonradial perturbations: Only the radial instability to collapse (or explosion) can remain. A turning-point criterion governs stability against collapse and is associated with upper and lower limits on the masses of relativistic stars, the analog for neutron stars of the Chandrasekhar limit. Finally, we consider the additional instabilities of rotating stars. These are nonaxisymmetric instabilities that radiate gravitational waves. They may set an upper limit on the spin of old neutron stars spun up by accretion and on nascent stars that form with rapid enough rotation. Chandrasekhar's was again the pioneering paper, showing that gravitational radiation can drive a nonaxisymmetric instability (Chandrasekhar 1970).

2. Action and canonical energy

One can obtain an action for stellar perturbations by introducing a Lagrangian displacement ξ^α joining each unperturbed fluid trajectory (the unperturbed worldline of a fluid element) to the corresponding trajectory of the perturbed fluid. We denote by p, ϵ, ρ and u^α the fluid's pressure, energy density, rest-mass density and 4-velocity, respectively. A perturbative description can be made precise by introducing a family of (time dependent) solutions

$$\mathcal{Q}(\lambda) = \{g_{\alpha\beta}(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda)\}, \tag{1}$$

and comparing to first order in λ the perturbed variables $Q(\lambda)$ with their equilibrium values $Q(0)$.

Eulerian and Lagrangian changes in the fluid variables are defined by

$$\delta\mathcal{Q} := \left.\frac{d}{d\lambda}Q(\lambda)\right|_{\lambda=0}, \qquad \Delta\mathcal{Q} = (\delta + \mathcal{L}_{\boldsymbol{\xi}})\mathcal{Q}, \tag{2}$$

with $\mathcal{L}_{\boldsymbol{\xi}}$ the Lie derivative along ξ^α.

Because oscillations of a neutron star proceed on a dynamical timescale, a timescale faster than that of heat flow, one requires that the Lagrangian change Δs in the entropy per unit rest mass vanishes, and perturbations of u^α, ρ and ϵ are expressed in terms of ξ^α and $h_{\alpha\beta} := \delta g_{\alpha\beta}$ by

$$\Delta u^\alpha = \frac{1}{2} u^\alpha u^\beta u^\gamma \Delta g_{\beta\gamma}, \qquad \Delta\rho = -\frac{1}{2}\rho q^{\alpha\beta}\Delta g_{\alpha\beta}, \qquad \Delta\epsilon = -\frac{1}{2}(\epsilon + p) q^{\alpha\beta}\Delta g_{\alpha\beta}, \tag{3}$$

with $\Delta g_{\alpha\beta} = h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$. Our restriction to adiabatic perturbations means that the Lagrangian perturbation in the pressure, Δp is given by

$$\frac{\Delta p}{p} = \Gamma \frac{\Delta\rho}{\rho} = -\frac{1}{2}\Gamma\, q^{\alpha\beta}\Delta g_{\alpha\beta}, \tag{4}$$

where the adiabatic index Γ is defined by

$$\Gamma = \frac{\partial \log p(\rho, s)}{\partial \log \rho} = \frac{\epsilon + p}{p}\frac{\partial\, p(\epsilon, s)}{\partial \epsilon}. \tag{5}$$

The perturbed Einstein-Euler equations,

$$\delta(G^{\alpha\beta} - 8\pi\, T^{\alpha\beta}) = 0, \tag{6}$$

are self-adjoint in the weak sense that they are a symmetric system up to a total divergence: For any pairs $(\xi^\alpha, h_{\alpha\beta})$ and $(\widehat{\xi}^\alpha, \widehat{h}_{\alpha\beta})$, the symmetry relation has the form

$$\widehat{\xi}_\beta \delta(\nabla_\gamma T^{\beta\gamma}\sqrt{|g|}) + \frac{1}{16\pi}\widehat{h}_{\beta\gamma}\delta\left[(G^{\beta\gamma} - 8\pi T^{\beta\gamma})\sqrt{|g|}\right] = -2\mathcal{L}(\widehat{\xi}, \widehat{h}; \xi, h) + \nabla_\beta \Theta^\beta, \tag{7}$$

where $\mathcal{L}$ is symmetric under interchange of (ξ, h) and $(\widehat{\xi}, \hat{h})$. A symmetry relation of the form (7) implies that $\mathcal{L}^{(2)}(\xi, h) := \frac{1}{2}\mathcal{L}(\xi, h; \xi, h)$ is a Lagrangian density and

$$I^{(2)} = \int d^4x \mathcal{L}^{(2)} \tag{8}$$

is an action for the perturbed system.

The conserved canonical energy is associated with the timelike Killing vector is the Hamiltonian of the perturbation, expressed in terms of configuration space variables,

$$E_c = \int_S d^3x\, \alpha(\Pi^\alpha \mathcal{L}_{\mathbf{t}}\xi_\alpha + \pi^{\alpha\beta}\mathcal{L}_{\mathbf{t}} h_{\alpha\beta} - \mathcal{L}^{(2)}), \tag{9}$$

where α is the lapse function, and Π^α and $\pi^{\alpha\beta}$ are the momenta conjugate to ξ^α and $h_{\alpha\beta}$,

$$\Pi^\alpha = -n_\gamma \Pi^{\gamma\alpha}, \qquad \pi^{\alpha\beta} = -n_\gamma \pi^{\gamma\alpha\beta}, \tag{10}$$

with

$$\Pi^{\alpha\beta} = \frac{1}{2}\frac{\partial \mathcal{L}(\xi, h; \xi, h)}{\partial \nabla_\alpha \xi_\beta}, \tag{11}$$

$$\pi^{\alpha\beta\gamma} = \frac{1}{2}\frac{\partial \mathcal{L}(\xi, h; \xi, h)}{\partial \nabla_\alpha h_{\beta\gamma}}. \tag{12}$$

The negative signs in Eq. (10) are associated with the choice of a future pointing unit normal and the signature $-+++$.

The corresponding canonical momentum has the form

$$J_c = \int_S d^3x\, \alpha(\Pi^\alpha \mathcal{L}_{\boldsymbol{\phi}} \xi_\alpha + \pi^{\alpha\beta}\mathcal{L}_{\boldsymbol{\phi}} h_{\alpha\beta}). \tag{13}$$

If one foliates the background spacetime by a family of spacelike but asymptotically null hypersurfaces, the difference $E_2 - E_1$ in E_c from one hypersurface to another to its future is the energy radiated in gravitational waves to future null infinity. Because this energy is positive definite, E_c can only decrease. This suggests that a condition for stability is that E_c be positive for all initial data.

This is, in fact, an appopriate stability criterion, but there is a subtlety, associated with a gauge freedom in choosing a Lagrangian displacement: There is a class of *trivial* displacements, for which the Eulerian changes in all fluid variables vanish. For a one (two) parameter equation of state, these correspond to rearranging fluid elements with the same value of ρ (and s).[2] For a trivial displacement η^α, the same physical perturbation is described by the pairs $h_{\alpha\beta}, \xi^\alpha$ and $h_{\alpha\beta}, \xi^\alpha + \eta^\alpha$, but the canonical energy is not invariant under addition of a trivial displacement, and its sign depends on this kind of gauge freedom. There is, however, a preferred class of *canonical* displacements, the displacements ξ^α that are orthogonal to all trivial displacements, with respect to the symplectic product of two perturbations,

$$W(\widehat{\xi}, \widehat{h}; \xi, h) := \int_\Sigma (\widehat{\Pi}_\alpha \xi^\alpha + \widehat{\pi}^{\alpha\beta} h_{\alpha\beta} - \Pi_\alpha \widehat{\xi}^a - \pi^{\alpha\beta}\widehat{h}_{\alpha\beta}) d^3x. \tag{14}$$

The criterion for stability can then be phrased as follows:

1. If $E < 0$ for some canonical data on Σ, then the configuration is unstable or marginally stable: There exist perturbations on a family of asymptotically null hypersurfaces Σ_u that do not die away in time.
2. If $E > 0$ for all canonical data on Σ, the magnitude of E is bounded in time and only finite energy can be radiated.

The trivial displacements are relabelings of fluid elements with the same baryon density and entropy per baryon. They are Noether-related to conservation of circulation in surfaces of constant entropy per baryon (Calkin 1963; Friedman & Schultz 1978), and canonical displacements

[2] This is not the gauge freedom associated with infinitesimal diffeos of the metric and matter, but a redundancy in the Lagrangian- displacement description of perturbations that is already present in a Newtonian context.

are displacements that preserve the circulation of each fluid ring – for which the Lagrangian change in the circulation vanishes.

For perturbations that are not spherical, stable perturbations have positive energy and die away in time; unstable perturbations have negative canonical energy and radiate negative energy to infinity, implying that E becomes increasingly negative. One would like to show that when $E < 0$ a perfect-fluid configuration is strictly unstable, that within the linearized theory the time-evolved data radiates infinite energy and that $|E|$ becomes infinite along a family Σ_u of asymptotically null hypersurfaces. There is no proof of this conjecture, but it is easy to see that if $E < 0$, the time derivatives $\dot{\xi}^\alpha$ and $\dot{h}_{\alpha\beta}$ must remain finitely large. Thus a configuration with $E < 0$ will be strictly unstable unless it admits nonaxisymmetric perturbations that are time dependent but *nonradiative*.

3. Local stability

The criterion for the stability of a spherical star against convection is easy to understand. When a fluid element is displaced upward, if its density decreases more rapidly than the density of the surrounding fluid, then the element will be buoyed upward and the star will be unstable. If, on the other hand, the fluid element expands less than its surroundings it will fall back, and the star will be stable to convection.

As this argument suggests, criteria for convective stability are *local*, involving perturbations restricted to an arbitrarily small region of the star or, for axisymmetric perturbations, to an arbitrarily thin ring. For local perturbations, the change in the gravitational field can be ignored: A perturbation in density of order $\delta\epsilon/\epsilon$ that is restricted to a region of volume $V \ll R^3$ (R the radius of the star) can be regarded as adding or subtracting from the source a mass δm of order $\delta\epsilon V$. Then

$$\frac{\delta m}{M} \sim \frac{V}{R^3}\frac{\delta\epsilon}{\epsilon} \ll \frac{\delta\epsilon}{\epsilon}. \tag{15}$$

The change in the metric is then also smaller than $\delta\epsilon/\epsilon$ by the factor V/R^3, arbitrarily small when the support of the matter perturbation is arbitrarily small. Note that, because the metric perturbation is gauge-dependent, this statement about the smallness of the metric is also gauge-dependent. A more precise way of stating this property of a local perturbation is that a gauge can be chosen in which the metric perturbation is smaller than the density perturbation by a factor of order V/R^3.

Convective instability of spherical relativistic stars was discussed by Thorne (1966) and subsequently, with greater rigor, by Kovetz (1967) and Schutz (1970). An initial heuristic treatment by Bardeen (1970) of convective instability of differentially rotating stars was made more precise and extended to models with heat flow and viscosity by Seguin (1975).

Consider a fluid element displaced radially outward from an initial position with radial coordinate r to $r + \xi$. The displacement vector then has components $\xi^\mu = \delta^\mu_r \xi$. The fluid element expands (or, if displaced inward, contracts), with its pressure adjusting immediately – in sound travel time across the fluid element – to the pressure outside:

$$\Delta p = \boldsymbol{\xi} \cdot \nabla p = \frac{dp}{dr}\xi. \tag{16}$$

Heat diffuses more slowly, and the analysis assumes that the motion is faster than the time for heat to flow into or out of the fluid element: The perturbation is *adiabatic*:

$$\begin{aligned} \Delta\epsilon &= \left(\frac{\partial\epsilon}{\partial p}\right)_s \Delta p \\ &= \left(\frac{\partial\epsilon}{\partial p}\right)_s \frac{dp}{dr}\xi = \Gamma\frac{\epsilon+p}{p}\frac{dp}{dr}\xi, \end{aligned} \tag{17}$$

where $\Gamma := \left(\dfrac{\partial \log p}{\partial \log \rho}\right)_s$ and we have used the adiabatic conditions (3) and (4).

The difference $\Delta_\star\epsilon$ in the density of the surrounding star between r and $r+\xi$ is given by

$$\Delta_\star\epsilon = \xi\frac{d\epsilon}{dr}. \tag{18}$$

The displaced fluid element falls back if $|\Delta\epsilon| < |\Delta_\star\epsilon|$ – if, that is, the fluid element's density decreases more slowly than the star's density:

$$\left(\frac{\partial p}{\partial\epsilon}\right)_s \left|\xi\frac{dp}{dr}\right| < \left|\xi\frac{d\epsilon}{dr}\right|. \tag{19}$$

The star is then stable against convection if the inequality,

$$\left(\frac{dp}{d\epsilon}\right)_\star := \frac{dp/dr}{d\epsilon/dr} < \left(\frac{\partial p}{\partial\epsilon}\right)_s, \tag{20}$$

is satisfied, unstable if the inequality is in the opposite direction.

The convective stability criterion can also be stated in terms of the temperature gradient: If the temperature gradient is superadiabatic – if T decreases faster than an adiabatically displaced fluid element – then the star is unstable against convection.

Within seconds after its formation, a neutron star cools to a temperature below the Fermi energy per nucleon, below 10^{12} K $\sim$ 100 MeV. Its neutrons and protons are then degenerate, with a nearly homentropic equation of state. The star is convectively stable, but its convection modes have low frequencies (of order 100 Hz or smaller). The nonzero frequency arises from the a composition gradient in the star, a changing ratio of neutrons to protons. A displaced fluid element does have time to adjust its composition to match that of the background star.

For spherical stars, any perturbation can be written as a superposition of spherical harmonics that are axisymmetric about some axis, and one therefore need only consider stability of axisymmetric perturbations. In fact, Detweiler & Ipser (1973) (generalizing a Newtonian result due to Lebovitz (1965)), show that, apart from local instability to convection, one need only consider radial perturbations: *If a nonrotationg star is stable to radial oscillations and stable against convection, the star is stable*. The Detweiler-Ipser argument shows that the Schwarzschild criterion (20) for stability against convection implies that there are no zero-frequency nonradial modes with polar parity, no time-independent polar-parity solutions to the perturbed Einstein-Euler system. The argument, by continuity of the frequency of outgoing modes, is compelling but not rigorous. It could be made more cleanly and without assumptions about normal modes if one could show directly that the canonical energy was always positive. This may follow from an integral inequality (associated with Eq. (42) of (Detweiler & Ipser 1973)), that is central to the Detweiler-Ipser argument. For a local perturbation – a perturbation for which the metric perturbation is negligible – the criterion for convective instability can easily be written in terms of the canonical energy E_c: For time-independent initial data with $\delta\epsilon = 0, \Delta\epsilon \neq 0$,

$$E_c = \int \frac{1}{\epsilon + p} \left[\left(\frac{\partial p}{\partial \epsilon} \right)_s - \left(\frac{dp}{d\epsilon} \right)_\star \right] \Delta\epsilon^2 \alpha dV, \tag{21}$$

and there are time-independent axisymmetric initial displacements ξ^α for which the canonical energy E_c of a rotating barotropic star is negative if and only if the generalized Schwarzschild criterion is violated.

3.1 Convective instability due to differential rotation: the Solberg criterion

Differentially rotating stars have one additional kind of convective (local) instability. If the angular momentum per unit rest mass, $j = hu_\alpha\phi^\alpha$, decreases outward from the axis of symmetry, the star is unstable to perturbations that change the differential rotation law.

The criterion is easy to understand in a Newtonian context. Consider a ring of fluid in the star's equatorial plane that is displaced outward from r to $r + \xi$, conserving angular momentum and mass. Again the displaced ring immediately adjusts its pressure to that of the surrounding star. If the ring's centripetal acceleration is larger that the net restoring force from gravity and the surrounding pressure gradient, it will continue to move outward. Now in the unperturbed star, the centripetal acceleration is equal to the restoring force. As in the discussion of convective instability, the displaced fluid element encounters the pressure gradient and gravitational field of the uperturbed star at its new position, and the restoring force is the restoring force on a fluid element at $r + \xi$ in the unperturbed star. Thus, if the displaced fluid ring has the same value of v^2/r as the surrounding fluid it will be in equilibrium, and the star will be marginally stable. If a displaced fluid ring has larger v^2/r than its surrounding fluid the star will be unstable.

The difference in acceleration for the background star is $\Delta_\star(v^2/r) = \xi^r \dfrac{d}{dr}(v^2/r)$, and stability then requires

$$\xi^r \frac{d}{dr}\left(\frac{v^2}{r}\right) - \Delta\frac{v^2}{r} > 0, \tag{22}$$

for $\xi^r > 0$.

Because $\Delta j = 0$ and $v(j,r) = j(r)/r$, we have

$$\Delta\frac{v^2}{r} = \Delta\frac{j^2}{r^3} = j^2\xi^r\frac{d}{dr}\frac{1}{r^3}, \tag{23}$$

while

$$\Delta_\star\frac{v^2}{r} = \xi^r\frac{d}{dr}\frac{j^2}{r^3}, \tag{24}$$

implying

$$\Delta_\star\frac{v^2}{r} - \Delta\frac{v^2}{r} = \xi^r\frac{1}{r^2}\frac{dj^2}{dr}; \tag{25}$$

and the star is stable only if $\dfrac{dj}{dr} > 0$ in the equatorial plane (for $j > 0$), or, equivalently, only if $\partial_\varpi(\varpi^2\Omega) > 0$.

For relativistic stars, the same criterion ordinarily holds, where the specific angular momentum $j = hu_\phi$ is the angular momentum per unit rest mass. Bardeen (1970) gives a heuristic argument for this criterion, and a subsequent comprehensive treatment, including heat flow and viscosity, is due to Seguin (1975). Abramowicz (2004) provides a much quicker and more intuitive derivation for a homentropic star with no dissipation. (The last paragraph was its Newtonian version.)

For a differentially rotating homentropic star with metric

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(dr^2 + r^2 d\theta^2), \tag{26}$$

the angular momentum per unit baryon mass is $\dfrac{\epsilon+p}{\rho}u_\phi = \dfrac{\epsilon+p}{\rho}\dfrac{e^\psi v}{\sqrt{1-v^2}}$, where $v = e^{\psi-\nu}(\Omega-\omega)$ is the fluid velocity measured by a zero-angular-momentum observer. The canonical energy of a local axisymmetric perturbation with $\delta p = 0$ is given by

$$E_c = \int \frac{(\epsilon+p)}{(1-v^2)^2}\left[2v\xi^\alpha\nabla_\alpha(\psi-\nu) - (1+v^2)e^{\psi-\nu}\xi^\alpha\nabla_\alpha\omega\right]\frac{\partial v}{\partial j}\xi^\alpha\nabla_\alpha j\,\sqrt{-g}d^3x, \tag{27}$$

implying that there are perturbations for which $E_c < 0$ unless

$$\xi^\alpha\nabla_\alpha j > 0, \quad \text{for } \xi^\alpha \text{ outward-directed}, \tag{28}$$

where outward-directed is defined by

$$\xi^\alpha\left[\nabla_\alpha(\psi-\nu) - \frac{(1+v^2)}{2v}e^{\psi-\nu}\nabla_\alpha\omega\right] > 0. \tag{29}$$

The derivation of the criterion is valid for dust (pressure-free fluid) or for a single particle in the geometry of a rotating star or black hole, where it implies that a circular orbit is stable if and only if j increases outward along the surrounding family of circular equatorial orbits.

This is a simplest example of the turning-point criterion governing axisymmetric stability: A point of marginal stability along a sequence of circular orbits of a particle is a point at which j is an extremum. The turning-point condition can be rephrased in terms of the particle's energy. For a particle of fixed rest mass, the difference in energy of adjacent orbits is related to the difference in its angular momentum by

$$\delta E = \Omega \delta J.$$

Then a point of marginal stability along a sequence of circular orbits of a particle of fixed baryon mass is a point at which its energy is an extremum.

4. Instability to collapse: Turning point criterion

For spherical stars in the Newtonian approximation, instability sets in when the matter becomes relativistic, when the adiabatic index Γ (more precisely, its pressure-weighted average) reaches the value 4/3 characteristic of zero rest mass particles. This quickly follows from the Newtonian form of the canonical energy for radial perturbations of a spherical star: For an initial radial displacement ξ, with $\partial_t \xi = 0$,

$$E_c = \int_0^R dr \left\{ \frac{4}{r} p' r^2 \xi^2 + \frac{1}{r^2} \Gamma p \left[(r^2 \xi)' \right]^2 \right\}. \tag{30}$$

Choosing as initial data $\xi = r$ gives

$$E_c = \int_0^R dr r^2 p \left(\Gamma - \frac{4}{3} \right), \tag{31}$$

implying instability for $\Gamma < 4/3$.

In the stronger gravity of general relativity, even models with the stiffest equation of state must be unstable to collapse for some value of $R/M > 9/8$, the ratio for the most relativistic model of uniform density. By (in effect) computing the relativistic canonical energy,

$$E_c = \int_0^R e^{\lambda+\nu} \left\{ \left[\frac{4}{r} p' - \frac{p'^2}{\epsilon + p} + 8\pi p(\epsilon + p) \right] r^2 \xi^2 + \frac{e^{3\lambda-\nu}}{r^2} \Gamma p \left[(e^{-\nu} r^2 \xi)' \right]^2 \right\}, \tag{32}$$

Chandrasekhar (1964) showed that the stronger gravity of the full theory gives a more stringent condition for stability: A star is unstable if

$$\Gamma < \frac{4}{3} + K \frac{M}{R}, \tag{33}$$

where K is a positive constant of order 1. Because a gas of photons has $\Gamma = 4/3$ and massive stars are radiation-dominated, the instability can be important for stars with $M/R \gg 1$ (Chandrasekhar 1964; Fowler 1966).

Turning point instability

The best-known instability result in general relativity is the statement that instability to collapse is implied by a point of maximum mass and maximum baryon mass, along a sequence of uniformly rotating barotropic models with fixed angular momentum. A formal symmetry in the way baryon mass and angular momentum occur in the first law implies that (as in the case of circular orbits) points of instability are also extrema of angular momentum along sequences of fixed baryon mass.

For dynamical oscillations of neutron stars, the adiabatic index does not coincide with the polytropic index, $\Gamma \neq \frac{d \log p(r)/dr}{d \log \rho/dr}$. Chandrasekhar's criterion locates the point of dynamical instability, if one uses the adiabatic index in the canonical energy. The turning point method locates a *secular* instability — an instability whose growth time is long compared to the typical dynamical time of stellar oscillations. For spherical stars, the turning-point instability proceeds on a time scale slow enough to accommodate the nuclear reactions and energy transfer that accompany the change to a nearby equilibrium. For rotating stars, the time scale must also be long enough to accommodate a transfer of angular momentum between fluid rings. That is, the growth rate of the instability is limited by the time required for viscosity to redistribute the star's angular momentum. For neutron stars, this is expected to be short, probably comparable to the spin-up time following a glitch, and certainly short compared to the lifetime of a pulsar or an accreting neutron star. For this reason, it is the secular instability that sets the upper and lower limits on the mass of spherical and uniformly rotating neutron stars.

Note that, if one considers perturbations conforming to the effective equation of state satisfied by the equilibrium star, then Chandrasekhar's canonical energy criterion coincides with the turning-point criterion for spherical stars. The turning point criterion, however, has a longer history. In their 1939 paper, Oppenheimer and Volkoff had already used it to locate the stable part of a sequence of model neutron stars; and Misner & Zapolsky (1964) noticed that, along a sequence of neutron star models, the configuration at which the functional E_c first becomes negative appeared to be the model with maximum mass. In each case, they used models in which the equilibrium configuration and its perturbations are governed by the same one-parameter equation of state. A turning-point method, due initially to Poincaré (1885), then implies that points at which the stability of a mode changes are extrema of the mass (Harrison et al. 1965). See Thorne (1967) for a review of the turning point method applied to spherical neutron stars and (Thorne 1978) for later references; a somewhat different treatment is given by Zel'dovich and Novikov (1971). The generalization of the turning point criterion to rapidly rotating stars, due to Friedman, Ipser, and Sorkin (see below) (1988), is based on a general turning-point theorem due to Sorkin (1981, 1982).

One can easily understand why the instability sets in at an extremum of the mass by looking at a radial mode of oscillation of a nonrotating star with an equation of state $p = p(\rho), \epsilon = \epsilon(\rho)$.

Along the sequence of spherical equilibria, a radial mode changes from stability to instability when its frequency σ changes from real to imaginary, with $\sigma = 0$ at the point of marginal stability. Now a zero-frequency mode is just a time-independent solution to the linearized Einstein-Euler equations - a perturbation from one equilibrium configuration to a nearby equilibrium with the same baryon number. From the first law of thermodynamics, a perturbation that keeps the star in equilibrium satisfies

$$\delta M = \frac{\mu}{u^t} dN, \tag{34}$$

with μ the chemical potential and N the number of baryons. The relation implies that, for a zero frequency perturbation involving no change in baryon number, the change δM in mass must vanish. This is the requirement that the mass is an extremum along the sequence of equilibria. Models on the *high-density* side of the maximum-mass instability point are unstable: Because the turning point is a star with maximum baryon number as well as maximum mass, there are models on opposite sides of the turning point with the same baryon number. Because μ/u^t is a decreasing function of central density, the model on the high-density side of the turning point has greater mass than the corresponding model with smaller central density.

At the minimum mass, it is the *low-density* side that is unstable: Because the mass is a minimum, the model on the low-density side of the turning point has greater mass than the corresponding model with the same baryon number on the high-density side.

The precise statement of the turning-point criterion is the following result:

Theorem (Friedman, Isper & Sorkin 1988). Consider a continuous sequence of uniformly rotating stellar models based on an equation of state of the form $p = p(\epsilon)$. Let λ be the sequence parameter and denote the derivative $d/d\lambda$ along the sequence by $(\dot{\ })$.
(i) Suppose that the total angular momentum is constant along the sequence and that there is a point λ_0 where $\dot{M} = 0$ and where $\mu > 0$, $(\dot{\mu}\dot{M})^{\cdot} \neq 0$. Then the part of the sequence for which $\dot{\mu}\dot{M} > 0$ is unstable for λ near λ_0.
(ii) Suppose that the total baryon mass M_0 is constant along the sequence and that there is a point λ_0 where $\dot{M} = 0$ and where $\Omega > 0$, $(\dot{\Omega}\dot{M})^{\cdot} \neq 0$. Then the part of the sequence for which $\dot{\Omega}\dot{M} > 0$ is unstable for λ near λ_0.

Friedman, Ipser & Sorkin (1988) point out the symmetry between M_0 and J that implies the maximum-J form of the theorem, and Cook, Shapiro & Teukolsky (1992) first use the theorem in this form.

For rotating stars, the turning point criterion is a sufficient condition for secular instability to collapse. In general, however, collapse can be expected to involve differential rotation, and the turning point identifies only nearby uniformly rotating configurations with lower energy. Rotating stars are therefore likely to be secularly unstable to collapse at densities slightly lower than the turning point density. The onset of secular instability to collapse is at or before the onset of dynamical instability along a sequence of uniformly rotating stars of fixed angular momentum, and recent work by Rezzolla, Katami and Yoshida (2011) appears to show that rapidly rotating stars can also be dynamically unstable to collapse just prior to the turning point.

Searches to determine the line of turning points have covered the set of models with sequences of constant rest mass M_0, extremizing J on each one, or vice versa. This is a computationally expensive procedure, and a more efficient way is summarized in the following corollary due to Jocelyn Read (Read et al. 2009):

Regard M_0 and J as functions on the two-dimensional space of equilibria. Turning points are the points at which ∇M_0 and ∇J are parallel. An equivalent statement of this criterion is that the wedge product of the gradients vanishes: $dM_0 \wedge dJ = 0$; or, with the space of equilibria embedded in a 3-dimensional space, $\nabla M_0 \times \nabla J = 0$. In particular, with the space of equilibria parametrized by the central energy density ϵ_c and axis ratio $\mathfrak{r} = r_p/r_e$, the turning points satisfy

$$\frac{\partial(M_0, J)}{\partial(\epsilon_c, \mathfrak{r})} \equiv \frac{\partial M_0}{\partial \epsilon_c}\frac{\partial J}{\partial \mathfrak{r}} - \frac{\partial J}{\partial \epsilon_c}\frac{\partial M_0}{+\partial \mathfrak{r}} = 0. \tag{35}$$

5. Nonaxisymmetric instabilities

Rapidly rotating stars and drops of water are unstable to a bar mode that leads to fission in the water drops and is likely to be the reason many stars in the Universe are in close binary systems. Galactic disks are unstable to nonaxisymmetric perturbations that lead to bars and to spiral structure. And a related instability of a variety of nonaxisymmetric modes, driven by gravitational waves, the Chandrasekhar-Friedman-Schutz (CFS) instability (Chandrasekhar 1970; Friedman & Schultz 1978; Friedman 1978), may limit the rotation of young neutron stars. The existence of this gravitational-wave driven instability in rotating stars was first found by Chandrasekhar (1970) in the case of the $l = 2$ mode in uniformly rotating, uniform density Maclaurin spheroids. Subsequently, Friedman and Schutz (1978) showed that all rotating self-gravitating perfect fluid configurations are generically unstable to the emission of gravitational waves. Along a sequence of stars, the instability sets in when the frequency of a nonaxisymmetric mode vanishes in the frame of an inertial observer at infinity, and such zero-frequency modes of rotating perfect-fluid stellar models are marginally stable.

This review begins with a discussion of the CFS instability for perfect-fluid models and then outlines the work that has been done to decide whether the instability is present in young neutron stars and in old neutron stars spun up by accretion. For very rapid rotation and for slower but highly differential rotation, nonaxisymmetric modes can be *dynamically unstable*, with growth times comparable to the period of a star's fundamental modes, and the review ends with a brief discussion of these related dynamical instabilities.

To understand the way the CFS instability arises, consider first a stable spherical star. All its modes have positive energy, and the sign of a mode's angular momentum J_c about an axis depends on whether the mode moves clockwise or counterclockwise around the star. That is, a mode with angular and time dependence of the form $\cos(m\phi - \sigma_0 t)e^{-\alpha_0 t}$, has positive angular momentum J_c about the z-axis if and only if the mode moves in a positive direction: $\dfrac{\sigma_0}{m}$ is positive. Because the wave moves in a positive direction relative to an observer at infinity, the star radiates positive angular momentum to infinity, and the mode is damped. Similarly, a mode with negative angular momentum has negative pattern speed $\dfrac{\sigma_0}{m}$ and radiates negative angular momentum to infinity, and the mode is again damped.

Now consider a slowly rotating star with a backward-moving mode, a mode that moves in a direction opposite to the star's rotation. Because a short-wavelength fluid mode (a mode with a Newtonian counterpart, not a w-mode) is essentially a wave in the fluid, the wave moves with nearly the same speed relative to a rotating observer that it had in the spherical star. That means that an observer at infinity sees the mode dragged forward by the fluid. The frequency σ_r seen in a rotating frame is the frequency associated with the ϕ coordinate $\phi_r = \phi - \Omega t$ of a rotating observer, $\sigma_r = \sigma - m\Omega$. Then

$$m\phi - \sigma t = m\phi_r - (\sigma + m\Omega)t = m\phi_r - \sigma_r t,$$

implying that the frequency seen by the rotating observer is

$$\sigma_r = \sigma - m\Omega. \tag{36}$$

For a slowly rotating star, $\sigma_r \approx \sigma_0$. When the star rotates with an angular velocity greater than $|\sigma_r/m|$, the backward-going mode is dragged *forward* relative to an observer at infinity:

$$\frac{\sigma}{m} = \frac{\sigma_r}{m} + \Omega \tag{37}$$

is positive.

Because the pattern speed σ/m is now positive, the mode radiates positive angular momentum to infinity. But the canonical angular momentum is still negative, because the mode is moving backward relative to the fluid: The angular momentum of the perturbed star is smaller than the angular momentum of the star without the backward-going mode. As the star radiates positive angular momentum to infinity, J_c becomes increasingly negative, implying that the amplitude of the mode grows in time: *Gravitational radiation now drives the mode instead of damping it.*

For large m or small σ_0, σ/m will be positive when $\Omega \approx |\sigma_0/m|$. This relation suggests two classes of modes that are unstable for arbitrarily slow rotation: Backward-moving modes with large values of m and modes with any m whose frequency is zero in a spherical star. Both classes of perturbations exist. The usual p-modes and g-modes have finite frequencies for a spherical star and are unstable for $\Omega \gtrsim \sigma_0/m$; and r-modes, which have zero frequency for a non-rotating barotropic star, are unstable for all values of m and Ω (that is, those r-modes are unstable that are backward-moving in the rotating frame of a slowly rotating star).

We have so far not mentioned the canonical energy, but our key criterion for the onset of instability is a negative E_c. If we ignore the imaginary part of the frequency, the change in the sign of E_c follows immediately from the relation $J_c = -\sigma_p E_c$. To take the imaginary part

$\mathrm{Im}\sigma = \alpha \neq 0$ of the frequency into account, we need to use the fact that energy is lost at a rate $\dot{E}_c \propto \dddot{Q}^2 \propto \sigma^6$ for quadrupole radiation, with $\dot{E}_c$ proportional to higher powers of σ for radiation into higher multipoles. Because E_c is quadratic in the perturbation, it is proportional to $e^{-2\alpha t}$, implying $\alpha \propto \sigma^6$. Thus $\alpha/\sigma \to 0$ as $\sigma \to 0$, implying that for a normal mode E_c changes sign when σ_p changes sign.

Although the argument we have given so far is heuristic, there is a precise form of the statement that a stable, backward-moving mode becomes unstable when it is dragged forward relative to an inertial observer (Friedman & Schultz 1978; Friedman & Stergioulas 2011).

Theorem. Consider an outgoing mode $(h_{\alpha\beta}(\lambda), \xi^\alpha(\lambda))$, that varies smoothly along a family of uniformly rotating perfect-fluid equilibria, labeled by λ. Assume that it has t and ϕ dependence of the form $e^{i(m\phi-\sigma t)}$, that $\sigma = \mathrm{Re}\{\sigma\}$ satisfies $\sigma/m - \Omega < 0$ for all λ, and that the sign of σ/m is negative for $\lambda < \lambda_0$ and positive for $\lambda > \lambda_0$. Then in a neighborhood of λ_0, $\alpha := \mathrm{Im}\{\sigma\} \leq 0$; and if the mode has at least one nonzero asymptotic multipole moment at future null infinity, the mode is unstable ($\alpha < 0$) for $\lambda > \lambda_0$.

A corresponding result that does not rely on existence or completeness of normal modes is the statement that one can always choose canonical initial data to make $E_c < 0$ (Friedman 1978; Friedman & Stergioulas 2011).

The growth time τ_{GR} of the instability of a perfect fluid star is governed by the rate $\left.\dfrac{dE}{dt}\right|_{\mathrm{GR}}$ at which energy is radiated in gravitational waves:

$$\frac{1}{\tau_{\mathrm{GR}}} = -\frac{1}{2E_c} \left.\frac{dE_c}{dt}\right|_{\mathrm{GR}}, \tag{38}$$

where (Thorne 1980)

$$\left.\frac{dE}{dt}\right|_{\mathrm{GR}} = -\sigma(\sigma + m\Omega) \sum_{l\geq 2} N_l \sigma^{2l} \left(|\delta D_{lm}|^2 + |\delta J_{lm}|^2\right), \tag{39}$$

where D_{lm} and J_{lm} are the asymptotically defined mass and current multipole moments of the perturbation and $N_l = \dfrac{4\pi(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}$ is, for low l, a constant of order unity. In the Newtonian limit,

$$\delta D_{lm} = \int \delta\rho\, r^l Y_{lm} d^3x. \tag{40}$$

For a star to be unstable, the growth time τ_{GR} must be shorter than the viscous damping time $\tau_{\mathrm{viscosity}}$ of the mode, and the implications of this are discussed below. In particular because the growth time is longer for larger l, only low multipoles can be unstable in neutron stars.

Modes with polar and axial parity

The spherical symmetry of a nonrotating star and its spacetime implies that perturbations can be labeled by fixed values l, m labeling an angular harmonic: The quantities $h_{\alpha\beta}, \xi^\alpha, \delta\rho, \delta\epsilon, \delta p, \delta s$ that describe a perturbation are all proportional to scalar, vector and tensor spherical harmonics constructed from Y_{lm}, and perturbations with different l, m values decouple. Similarly, because spherical stars are invariant under parity (a map of each point P of spacetime to the diametrically opposite point on the symmetry sphere through P), perturbations with different parity decouple, the parity of a perturbation is conserved, and normal modes have definite parity. Perturbations associated with an l, m angular harmonic are said to have *polar* parity if they have the same parity as the function Y_{lm}, $(-1)^l$. Perturbations having parity $(-1)^{l+1}$, opposite to that of Y_{lm} have axial parity. In the Newtonian literature, modes of a rotating star that are continuously related to polar modes of a spherical star are commonly called *spheroidal*; while modes whose spherical limit is axial are called *toroidal*.

Every rotational scalar – ϵ, p, ρ, and the components of the perturbed metric $h_{\alpha\beta}$ and the perturbed fluid velocity δu^α in the t-r subspace – can be expressed as a superposition of scalar spherical harmonics $Y_{\ell m}$. As a result, modes of spherical stars that involve changes in any scalar are polar. On the other hand, the angular components of velocity perturbations can have either polar parity, with

$$\delta v = f(r)\nabla Y_{lm} \tag{41}$$

or axial parity, with Newtonian form

$$\delta v = f(r)\mathbf{r} \times \nabla Y_{lm}, \tag{42}$$

and the relativistic form $\delta u^\alpha \propto \epsilon^{\alpha\beta\gamma\delta}\nabla_\beta t \nabla_\gamma r \nabla_\delta Y_{lm}$.

There are two families of polar modes of perfect-fluid Newtonian stars, p-modes (pressure modes) and g-modes (gravity modes). For short wavelengths, the p-modes are sound waves, with pressure providing the restoring force and frequencies

$$\sigma = c_s k, \tag{43}$$

where k is the wavenumber and c_s is the speed of sound. The short-wavelength g-modes are modes whose restoring force is buoyancy, and their frequencies are proportional to the Brunt-Väisälä frequency, related to the difference between $dp/d\epsilon$ in the star and $c_s^2 = \partial p(\epsilon, s)/\partial\epsilon$. The fundamental modes of oscillation of a star (f-modes), with no radial nodes, can be regarded as a bridge between g-modes and p-modes.

Because axial perturbations of a spherical star involve no change in density or pressure, there is no restoring force in the linearized Euler equation, and the linear perturbation is a time-independent velocity field – a zero-frequency mode.[3] In a rotating star, the axial modes acquire

[3] Axial perturbations of the spacetime of a spherical star include both axial perturbations of the fluid and gravitational waves with axial parity. The axial-parity waves do not couple to the fluid perturbation, which is stationary in the sense that $\partial_t \delta u_\alpha = 0$.

a nonzero frequency proportional to the star's angular velocity Ω, a frequency whose Newtonian limit has the simple form

$$\sigma = \frac{(l-1)(l+2)}{l(l+1)} m\Omega, \tag{44}$$

where the harmonic time and angular dependence of the mode is $e^{i(m\phi - \sigma t)}$. These modes are called r-modes, their name derived from the Rossby waves of oceans and planetary atmospheres. The term r-mode can be usefully regarded as a mnemonic for a *rotationally restored* mode. Eq. (36) implies that the r-mode associated with every nonaxisymmetric multipole obeys the instability condition for every value of Ω: It is forward moving in an inertial frame and backwards moving relative to a rotating observer:

$$\sigma_r = -\frac{2m}{l(l+1)}\Omega, \tag{45}$$

with sign opposite to that of σ and m. Because the rate at which energy is radiated is greatest for the $l = m = 2$ r-mode, that is the mode whose instability grows most quickly and which determines whether an axial-parity instability can outpace viscous damping.

The instability of low-multipole r-modes for arbitrarily slow rotation is strikingly different from the behavior of the low-multipole f- and p-modes, which are unstable only for large values of Ω. The reason is that the frequencies of f- and p-modes are high, and, from Eq. (37), a correspondingly high angular velocity is needed before a mode that moves backward relative to the star is dragged forward relative to an inertial observer at infinity. Of the polar modes, f-modes with $l = m$ have the fastest growth rates; their instability points for uniformly rotating relativistic stars, found by Stergioulas (Friedman & Stergioulas 2011), are shown in Figure 1. (Work on these stability points of relativistic stars is reported in (Stergioulas & Friedman 1998; Yoshida & Eriguchi 1997; Yoshida & Eriguchi 1999; Zink et al. 2010; Gaertig et al. 2011)

The figure shows that, for uniform rotation, the $l = m = 2$ f-mode is unstable only for stars with high central density and therefore with masses greater than 1.4 $M_\odot$. Neutron stars, however, rotate differentially at birth, and the $l = 2$ mode, as well as well as f-modes with $l \geq 5$, could be initially unstable.

Implications of the instability

The nonaxisymmetric instability may limit the rotation of nascent neutron stars and of old neutron stars spun up by accretion; and the gravitational waves emitted by unstable modes may be observable by gravitational wave detectors. Whether a limit on spin is in fact enforced depends on whether the instability of perfect-fluid models implies an instability of neutron stars; and the observability of gravitational waves also requires a minimum amplitude and persistence of an unstable mode. We briefly review observational support for an instability-enforced upper limit on spin and then turn to the open theoretical issues.

Evidence for an upper limit on neutron-star spin smaller than the Keplerian frequency Ω_K comes from nearly 30 years of observations of neutron stars with millisecond periods, seen as

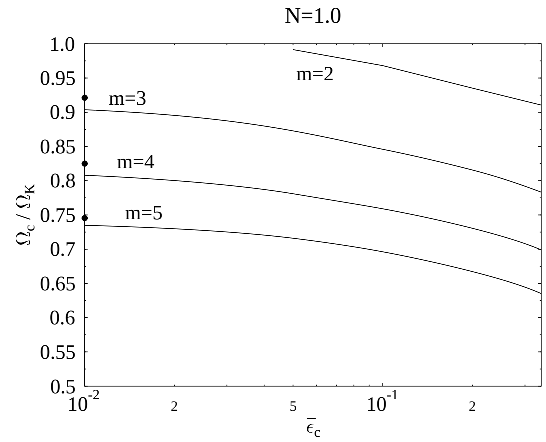

Figure 1. Critical angular velocity Ω/Ω_K vs. the dimensionless central energy density $\bar{\epsilon}_c$ for the $m = 2$, 3, 4 and 5 neutral modes of $N = 1.0$ polytropes. The filled circles on the vertical axis are the Newtonian values of the neutral points for each mode.

pulsars and as X-ray binaries. The observations reveal rotational frequencies ranging upward to 716 Hz and densely populating a range of frequencies below that. Selection biases against detection of the fastest millisecond radio pulsars have made conclusions about an upper limit on spin uncertain, but Chakrabarty argues that the class of sources whose pulses are seen in nuclear bursts (nuclear powered accreting millisecond X-ray pulsars) constite a sample without significant bias (Chakrabarty 2008); their distribution of spins is shown in Fig. 1 of that paper, reproduced as Fig. 2 below.

Summarizing his analysis, Chakrabarty writes, "There is a sharp cutoff in the population for spins above 730 Hz. RXTE has no significant selection biases against detecting oscillations as fast as 2000 Hz, making the absence of fast rotators extremely statistically significant." Even for a $1.4M_\odot$ star, 800 Hz is well below Ω_K for all but the stiffest candidate equations of state, and accreting pulsars are likely to have larger masses and still higher values of Ω_K.

A magnetic field of order 10^8 G can limit the spin of an accreting millisecond pulsar. Because matter within the magnetosphere corotates with the star, only matter that accretes from outside the magnetosphere can spin up the star, leading to an equilibrium period given approximately by (Ghosh & Lamb 1979)

$$P_{\rm eq} \sim \left(\frac{B}{10^{12}{\rm G}}\right)^{6/7} \left(\frac{\dot{M}}{10^{-9}M_\odot{\rm yr}^{-1}}\right)^{-3/7} . \qquad (46)$$

Because this period depends on the magnetic field, a sharp cutoff in the frequency of accreting stars is not an obvious prediction of magnetically limited spins; and a cutoff at a rotation rate of order 700-800 Hz is not consistent with a range of magnetic field strengths presumed to extend below 10^8 G.

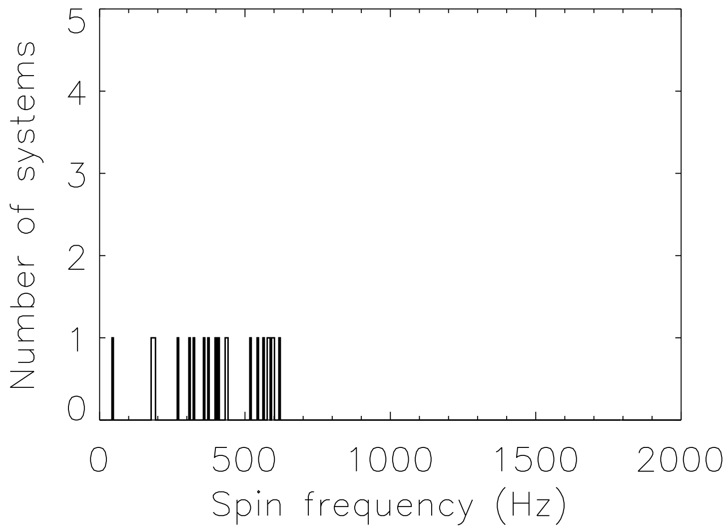

Figure 2. The spin frequency distribution of accreting millisecond X-ray pulsars. (From Chakrabarty 2008.)

Under what circumstances the CFS instability could limit the spin of recycled pulsars has now been studied in a large number of papers. References to this work can be found in the treatment in (Friedman & Stergioulas 2011) on which the present review is based and in comprehensive earlier discussions by Stergioulas (2003), by Andersson and Kokkotas (2001), and by Kokkotas and Ruoff (2001, 2002) briefer reviews of more recent work are given in (Andersson et al 2011; Owen 2010). References in the present review are generally limited to initial work and to a late paper that contains intervening references.

Whether the instability survives the complex physics of a real neutron star has been the focus of most recent work, but it remains an open question. Studies have focused on:

- Dissipation from bulk and shear viscosity and mutual friction in a superfluid interior;
- magnetic field wind-up;
- nonlinear evolution and the saturation amplitude; and
- the possiblity that a continuous spectrum replaces r-modes in relativistic stars.

We discuss these in turn and then summarize the implications for nascent, rapidly rotating stars and for old stars spun up by accretions.

Viscosity

When viscosity is included, the growth-time or damping time τ of an oscillation has the form

$$\frac{1}{\tau} = \frac{1}{\tau_{GR}} + \frac{1}{\tau_b} + \frac{1}{\tau_s}, \tag{47}$$

with τ_b and τ_s the damping times due to bulk and shear viscosity. Bulk viscosity is large at high temperatures, shear viscosity at low temperatures. This leaves a window of opportunity in which a star with large enough angular velocity can be unstable. The window for the $l = m = 2$ r-mode is shown in Fig. 3, for a representative computation of viscosity. The highest solid curves on left and right mark the critical angular velocity Ω_c above which the $l = m = 2$ r-mode is unstable. The curves on the left, show the effect of shear viscosity at low temperature, allowing instability when $\Omega < \Omega_K$ only for $T > 10^6$K; the curve on the right shows the corresponding effect of bulk viscosity, cutting off the instability at temperatures above about 4×10^{10}K.

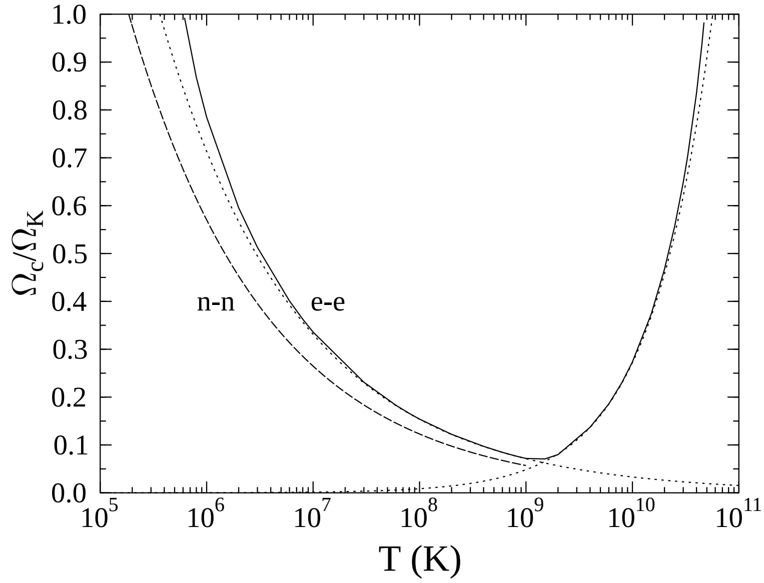

Figure 3. Critical angular velocity for the onset of the r-mode instability as a function of temperature (for a 1.5 $M_\odot$ neutron star model). The solid line corresponds to the $O(\Omega^2)$ result using electron-electron shear viscosity, and modified URCA bulk viscosity. The dashed line corresponds to the case of neutron-neutron shear viscosity. Dotted lines are $O(\Omega)$ approximations.

There is substantial uncertainty in the positions of both of these curves.

Bulk viscosity arises from nuclear reactions driven by the changing density of an oscillating fluid element, with neutrons decaying, $n \to p + e + \bar{\nu}_e$, as the fluid element expands and protons capturing electrons, $p + e \to n + \nu_e$, as it contracts. The neutrinos leave the star, draining energy from the mode. The rates of these *URCA* reactions increase rapidly with temperature and are fast enough to be important above about 10^9K, with an expected damping time τ_b given by

$$\frac{1}{\tau_{\rm b}} = \frac{1}{2E_c} \int \zeta(\delta\theta)^2 d^3x, \tag{48}$$

where $\theta = \nabla_\alpha u^\alpha$ is the divergence of the fluid velocity and the coefficient of bulk viscosity ζ is given by (Cutler, Lindblom & Splinter 1990)

$$\zeta = 6 \times 10^{25} \rho_{15}^2 T_9^6 \left(\frac{\omega_r}{1\mathrm{Hz}}\right)^{-2} \ \mathrm{g\,cm^{-1}\,s^{-1}}, \tag{49}$$

where $T_9 = T/(10^9\mathrm{K})$ With these values, bulk viscosity kills the instability in all modes above a few times 10^{10}K (Ipser & Lindblom 1991 a, b; Yoshida & Eriguchi 1995).

These equations and Fig. 3 assume that only *modified URCA* reactions can occur, that the URCA reactions require a collision to conserve four-momentum, and this will be true when the proton fraction is less than about $1/9$. If the equation of state turns out to be unexpectedly soft (and the mass is large enough), direct URCA reactions would be allowed, suppressing the instability for uniformly rotating stars at roughly 10^9K (Zdunik 1996). A soft equation of state is also more likely to lead to stars with hyperons in their core with an additional set of nuclear reactions that dissipate energy and increase the bulk viscosity (Jones 2010; Lindblom & Owen 2002; Haensel, Levenfish & Yakovlev 2002; Nayyar & Owen 2006; Haskell & Andersson 2010) or quarks (Madsen 1998; Madsen 2000; Andersson et al, 2002; Jaikumar et al 2008; Rupak & Jaikumar 2010).

In contrast to bulk viscosity, shear viscosity increases as the temperature drops. In terms of the shear tensor $\sigma_{\alpha\beta} = (\delta_\alpha^\gamma + u_\alpha u^\gamma)(\delta_\beta^\delta + u_\beta u^\delta)(\nabla_\gamma u_\delta + \nabla_\delta u_\gamma - \frac{2}{3} g_{\gamma\delta} \nabla_\epsilon u^\epsilon)$, the damping time is given by

$$\frac{1}{\tau_\mathrm{s}} = \frac{1}{E_c} \int \eta \delta\sigma^{\alpha\beta} \delta\sigma_{\alpha\beta}\, d^3x, \tag{50}$$

where η is the coefficient of shear viscosity. For nascent neutron stars hotter than the superfluid transition temperature (about 10^9K), the neutron-neutron shear viscosity coefficient is (Flowers & Itoh 1976)

$$\eta_n = 2 \times 10^{18} \rho_{15}^{9/4} T_9^{-2} \ \mathrm{g\,cm^{-1}\,s^{-1}}, \tag{51}$$

where $\rho_{15} = \rho/(10^{15}\mathrm{g\,cm^{-3}})$. Below the superfluid transition temperature, electron-electron scattering determines the shear viscosity in the superfluid core, giving (Cutler & Lindblom 1987)

$$\eta_e = 6 \times 10^{18} \rho_{15}^2 T_9^{-2} \ \mathrm{g\,cm^{-1}\,s^{-1}}. \tag{52}$$

Shear viscosity may be greatly enhanced after formation of the crust in a boundary layer (Ekman layer) between crust and core (Ushomirsky & Bildsten 1998; Lindblom et al 2000; Anderson et al 2000; Glampedakis & Anderson 2006a, Glampedakis & Anderson 2006b). The enhancement depends on the extent to which the core participates in the oscillation, parametrized by the slippage at the boundary. The uncertainty in this slippage appears to be the greatest current uncertainty in dissipation of the mode by shear viscosity, and it significantly affects the critical angular velocity of the r-mode instability in accreting neutron stars.

For f-modes, the part of the instability window in Fig. 3 to the left of 10^9 K is thought to be removed by another dissipative mechanism that comes into play below the superfluid transition temperature. Called mutual friction, it arises from the scattering of electrons off magnetized neutron vortices. Work by Lindblom and Mendell (1995) shows that mutual friction in the superfluid core completely suppresses f- and p-mode instabilities below the transition temperature. For the r-mode instability, subsequent work by the same authors (2000) finds that the mutual friction is much smaller, with a damping time of order 10^4 s, too long to be important.

In a recent paper, Gaertig et al. point out the possibility of an interaction between vortices and quantized flux tubes that would result in a much smaller value for the mutual friction. They argue that the resulting uncertainty is great enough that shear viscosity could be the dominant dissipative mechanism for f-modes as well as r-modes.

Magnetic field windup

At second-order in the perturbation, the nonlinear evolution of an unstable mode includes an axisymmetric part that describes a growing differential rotation. Because differential rotation will wind up magnetic field lines, the mode's energy could be transferred to the star's magnetic field (Spruit 1999; Rezzolla et al. 2000; Rezzolla et al. 2001b; Rezzolla et al 2001a; Cuofano& Drago 2010). Again there is large uncertainty about the strength of a toroidal magnetic field that will be generated by the differential rotation, what magnetic instabilities will arise, and what the effective dissipation will be. Apart from the studies cited here (all of which deal with r-modes) nearly all the remaining work on the evolution of unstable modes ignores magnetic fields.

Relativistic r-modes and a possible continuous spectrum

Relativistic r-modes have been computed by a number of authors (Kojima 1998; Kojima & Hosonuma 1999; Kojima & Hosonuma 2000; Lockitch, Andersson & Friedman 2001; Lockitch, Friedman & Andersson 2003; Lockitch, Andersson & Watts 2004; Andersson 1998; Ruoff & Kokkotas 2001; Ruoff & Kokkotas 2002; Ruoff, Stavridis & Kokkotas 2003; Kokkotas & Ruoff 2002; Yoshida & Lee 2002; Kastaun 2008). Where the Newtonian approximation has purely axial $l = m$ r-modes for barotropic stars at lowest order in Ω, in the full theory all rotationally restored modes include a polar part. The change in the structure of the computed r-modes are small, but that may not be the end of the story.

For non-barotropic stars Kojima found a single second-order eigenvalue equation for the frequency, to lowest nonvanishing order in Ω. The coefficient of the highest derivative term in that equation vanishes at some value of the radial coordinate r, for typical candidate neutron-star equations of state, and that singular behavior gives a continuous spectrum. Lockitch, Andersson & Watts (2004) consider the question of the continuous spectrum and the existence of r-modes in some detail. They argue that the singularity in the Kojima equation is an artifact of the slow-rotation approximation and is not present if one includes terms of order Ω^2. Their work is a strong argument for the existence of r-modes in non-barotropic models.

Showing the existence of the mode, however, does not decide the question of whether a continuous spectrum is also present or whether the existence of a continuous or nearly continuous spectrum significantly alters the evolution of an initial perturbation.

Nonlinear evolution

Linear perturbation theory is valid only for small-amplitude oscillations; as the amplitude of an unstable mode grows, couplings to other modes become increasingly important, and the mode ultimately reaches a saturation amplitude or is disrupted, losing coherence. The first nonlinear studies of the r-mode instability involved fully nonlinear 3+1 evolutions in which the r-mode was set at a large initial amplitude (Stergioulas & Font 2001) or was driven to large amplitude by an artificially large gravitational-radiation reaction term (Lindblom, Tohline & Vallisneri 2001, Lindblom, Tohline & Vallisneri 2002). On a few tens of dynamical timescales, saturation was seen only at an amplitude of order unity. Subsequently, simulations on longer timescales showed a coupling to daughter modes (Gressman et al. 2002; Lin & Suen 2006), suggesting that the actual saturation amplitude of the r-mode is smaller than the amplitude at which gravitational-radiation reaction was switched off in the short-timescale simulations.

The resolution of 3+1 simulations, however, is too low to see couplings to short-wavelength modes, and they cannot run for a time long enough to see the growth from a realistic radiation-reaction term. The alternative is to examine the nonlinear evolution in the context of higher-order perturbation theory. To do this, the Cornell group (initially with S. Morsink) (Arras et al. 2003; Schenk et al. 2002; Morsink 2002) constructed a second-order perturbation theory for rotating Newtonian stars, and then used the formalism to study the nonlinear evolution of an unstable r-mode. Their series of papers leaves little doubt that nonlinear couplings sharply limit the amplitude of an unstable r-mode, with a possible range of 10^{-1}-10^{-5} (see (Bondarescu, Teukolsky & Wasserman 2007) and references therein).

The nonlinear development of the f-mode instability has been modeled in three-dimensional, hydrodynamical simulations (in a Newtonian framework) by Ou, Tohline & Lindblom (2004) and by Shibata & Karino (2004), essentially confirming previous approximate results obtained in (Lai & Shapiro 1995). Kastaun et al. (2010) report an initial nonlinear study of f-modes in general relativity. In the framework of a 3+1 simulation in a Cowling approximation (a fixed background metric of the unperturbed rotating star), they find limits on the amplitude of less than 0.1, set by wave-breaking and by coupling to inertial modes. This can be regarded as an upper limit on the amplitude, with second-order perturbative computations still to be done.

Instability scenarios in nascent neutron stars and in old accreting stars

Both r-modes and f-modes may be unstable in nascent neutron stars that are rapidly rotating at birth. Recent work on f-modes in relativistic models (Gaertig et al.; Gaertig & Kokkotas 2010) finds growth times substantially shorter than previously computed Newtonian values. In particular, the $l = m = 3$ and $l = m = 4$ f-modes have growth times of 10^3-10^5 s for Ω near Ω_K. In a typical scenario, a star with rotation near the Kepler limit becomes unstable within a minute of formation, when the temperature has dropped below 10^{11}K. As the temperature drops further, the instability grows to saturation amplitude in days or weeks. Loss of angular momentum to gravitational waves spins down the star until the critical angular velocity is reached below which the star is stable, at or before the time at which the core becomes a superfluid. The $l = m = 3$ mode could be a source of observable gravitational waves for supernovae in or near the Galaxy.

The time over which the instability is active depends on the saturation amplitude, the cooling rate, and the superfluid transition temperature, and all of these have large uncertainties. The time at which a superfluid transition occurs could be shorter than a year, but recent analyses of the cooling of a neutron star in Cassiopeia A (Page et al. 2011; Shternin et al. 2011) suggest a superfluid transition time for that star of order 100 years.

The scenario for the $l = m = 2$ r-mode instability of a nascent star is similar. The r-mode instability itself was pointed out by Andersson (1998), with a mode-independent proof for relativistic stars given by Friedman and Morsink (1998). First computations of the growth and evolution were reported by Lindblom et al. (1998) and Andersson et al. (1999), with effects of a crust discussed in Lindblom et al. (2000). Intervening work is referred to in a recent paper by Bondarescu et al. (2008); the simulations reported by Bondarescu et al. include nonlinear couplings that saturate the amplitude and the alternative possibilities for viscosity that we have discussed above. The r-mode's saturation amplitude is likely to be lower than that of the f-modes, and it is likely to persist longer because of its low mutual friction.

As mentioned above, the r-mode instability of neutron stars spun up by accretion has been more intensively studied in connection with the observed spins of LMXBs. Papaloizou & Pringle (1978) suggested the possibility of accretion spinning up a star until it becomes unstable to the emission of gravitational waves and reaches a steady state, with the angular momentum gained by accretion equal to the angular momentum lost to gravitational waves. Following the discovery of the first millisecond pulsar, Wagoner examined the mechanism in detail for CFS unstable f-modes (Wagoner 2002). Although mutual friction appears to rule out the steady-state picture for f-modes, it remains a possibility for r-modes (Bildsten 1998; Andersson et al 1999; Andersson et al. 2000; Wagoner 2002). Levin (1999) and (independently) Spruit (1999), however, pointed out that viscous heating of the neutron star by its unstable oscillations will lower the shear viscosity and so increase the mode's growth rate, leading to a runaway instability. The resulting scenario is a cycle in which a cold, stable neutron star is spun up over a few million years until it becomes unstable; the star then heats up, the instability grows, and the star spins down until it is again stable, all within a few months; the star then cools, and the cycle repeats.

This scenario would rule out r-modes in LMXBs as a source of detectable gravitational waves because the stars would radiate for only a small fraction of the cycle. A small saturation amplitude, however, lengthens the time spent in the cycle, possibly allowing observability (Heyl 2002). The steady state itself remains a possible alternative in stars whose core contains hyperons or free quarks (or if the "neutron stars" are really strange quark stars) (Andersson et al 2002; Lindblom & Owen 2002; Wagoner 2002; Reisenegger & Bonačić 2003; Nayyar & Owen 2006; Haskell & Andersson 2010). Heating the core increases the bulk viscosity, and with an exotic core, this growth in the bulk viscosity is large enough to prevent the thermal runaway and allow a steady state. Recent work by Bondarescu et al. (2007) constructs nonlinear evolutions (restricted to 3 coupled modes) that include neutrino cooling, shear viscosity, hyperon bulk viscosity and dissipation at the core-crust boundary layer, with parameters to span a range of uncertainty in these various quantities. They display the regions of parameter space associated with the alternative scenarios just outlined – steady state, cycle, and fast and slow runaways. In all cases, the r-mode amplitude remains very small ($\sim 10^{-5}$), but because of the long duration of the instability, such systems are still good candidates for gravitational wave detection by advanced LIGO class interferometers (Bondarescu et al 2007; Watts & Krishnan 2009; Owen 2010).

Dynamical nonaxisymmetric instability

Work on dynamical nonaxisymmetric instabilities is largely outside the scope of this review. They are most likely to be relevant to protoneutron stars and to the short-lived hypermassive neutron stars that form in the merger of a double neutron star system. Unless the star has unusually high differential rotation, instability requires a large value of the ratio $T/|W|$ of rotational kinetic energy to gravitational binding energy: comparable to the value $T/|W| = 0.27$ that marks the dynamical instability of the $l = m = 2$ mode of uniformly rotating uniform density Newtonian models (the Maclauring spheroids). This bar instability, if present, will emit strong gravitational waves with frequencies in the kHz regime. The development of the instability and the resulting waveform have been computed numerically in the context of both Newtonian gravity and in full general relativity (see (Houser et al. 1994; Tohline et al. 1985; Shibata et al. 2000; Manca et al. 2007) for representative studies).

Uniformly rotating neutron stars have maximum values of $T/|W|$ smaller than 0.14, apparently precluding dynamical nonaxisymmetric instability. For highly differential rotation, however, Centrella et al. (2001) found a one-armed ($m = 1$) instability for smaller rotation, for $T/|W| \sim 0.14$, but for a polytropic index of $N = 3$ which is not representative for neutron stars. Remarkably, Shibata et al. (2002, 2003) then found found an $m = 2$ instability for $T/|W|$ as low as 0.01, for models with polytropic index $N = 1$, representing a stiffness appropriate to neutron stars. These instabilities appear to be related to the existence of corotation points, where the pattern speed of the mode matches the star's angular velocity (Watts, Anderson & Jones 2005; Saijo & Yoshida 2006); Ou and Tohline tie the growth of the instability to a resonant cavity associated with a minimum in the vorticity to density ratio (the so-called vortensity) (Ou & Tohline 2006). Collapsing cores in supernovae are differentially rotating, and these instabilities of proto-neutron stars arise in simulations of rotating core collapse (Ott et al. 2005; Ott 2009). Because they can radiate more energy in gravitational waves than the post-bounce burst signal itself, interest in these dynamical instabilities is strong.

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