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Non-linear stability of L_4 in the restricted three body problem for radiated axes symmetric primaries with resonances

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Abstract. We have investigated the non-linear stability of the triangular libration point L_4 of the Restricted three body problem under the presence of the third and fourth order resonances, when the bigger primary is an oblate body and the smaller a triaxial body and both are source of radiation. It is found through Markeev's theorem that L_4 is always unstable in the third order resonance case and stable or unstable in the fourth order resonance case depending upon the values of the parameters A_1, A_1', A_2', P and P', where A_1, A_1' and A_2', depends upon the lengths of the semi axes of the primaries and P and P' are the radiation parameters.

Keywords : Restricted three body problem, axis symmetric body, libration points, non-linear stability, Markeev's theorem.

1. Introduction

In the present paper, our aim is to investigate the non-linear stability of the triangular libration point L_4, under the presence of resonances in the restricted three body problem when the bigger primary is an oblate body and the smaller a triaxial body and both are sources of radiation and their equatorial planes are coincident with the plane of motion. Hallan *et al.* (2000) studied the same model in the absence of the resonances. For this they applied the Moser's modified version of Arnold's theorem (1961).

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Arnold proved that if

(i) $k_1\omega_1 + k_2\omega_2 \neq 0$ for all pairs (k_1, k_2) of rational integers, where ω_1, ω_2 are the basic frequencies for the linear dynamical system, and

(ii) Determinant $D \neq 0$,

where

$$\begin{aligned} D &= \det(b_{ij}) \qquad (i,j = 1,2,3), \\ b_{ij} &= \left(\frac{\partial^2 H}{\partial I_i \partial I_j}\right)_{I_i = I_j = 0} \qquad (i,j = 1,2), \\ b_{i3} &= b_{3i} = \left(\frac{\partial H}{\partial I_i}\right)_{I_i = I_j = 0} \qquad (i,j = 1,2), \\ b_{33} &= 0 \quad \text{and} \quad H = \omega_1 I_1 + \omega_2 I_2 + \frac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \dots, \end{aligned}$$

H is the normalized Hamiltonian with I_1 and I_2 as the action momenta co-ordinates, then on each energy manifold $H = h$ in the neighborhood of equilibrium, there exits invariant tori of quasi periodic motion which divide the manifold and consequently the equilibrium is stable. This is valid for a system with two degrees of freedom, which is the case under consideration. Moser has shown that Arnold's theorem is true if the condition (i) of the theorem is replaced by $k_1\omega_1 + k_2\omega_2$ for all pairs (k_1, k_2) of rational integers such that $|k_1| + |k_2| \leq 4$. They found that L_4 is stable for $0 < \mu < \mu_c$, (μ_c = a critical value of μ) in the non-linear sense except at three mass parameters μ_1, μ_2 and μ_3 where Moser's theorem is not applicable. Here μ_1 corresponds to the resonance case $\omega_1 = 2\omega_2$ and μ_2 to the resonance case $\omega_1 = 3\omega_2$ (Hallan *et al.* 2000). We may note that Moser's condition (i) is not satisfied at these values.

As far as resonance cases are concerned, they have been studied by Henrard (1970), Markeev (1978), Kunitsyn and Perezhogin (1986), Chaudhary (1987, 1988), Thakur and Singh (1997), Gozdziewski and Maciejewski *et al* (1998), and Chandra Naveen (2004).

In all the above studies, the case when the bigger primary is an oblate body and the smaller a triaxial body and both are sources of radiation, have not been considered. We have investigated the non-linear stability of the triangular libration point L_4, with the help of Markeev's (1978) theorem for the resonance cases $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$. In order to apply Markeev's theorem we have to compute Birkhoff's normal form upto the fourth order terms of the Hamiltonian. The normal form of the Hamiltonian contains resonance terms for both the resonance cases $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$.

The original version of Markeev's theorems is in Russian. Many authors have used the translated English version of Markeev's theorem for stability of the triangular libration point L_4 in the restricted problem which states as follows:

Markeev's Theorem (Translated version).

For $\omega_1 = 2\omega_2$.

With the suitable choice of the variables q_i, p_i in the case $\omega_1 = 2\omega_2$ the Hamiltonian $H = H_2 + H_3 + H_4 + \ldots$ reduces to

$$\begin{aligned} H &= 2\omega_2 r_1 - \omega_2 r_2 - r_2\sqrt{r_1}\sqrt{(x_{1002}^2 + y_{1002}^2)\omega_2} \\ &\quad \times \sin(\phi_1 + \phi_2) + o((r_1 + r_2)^2)\,. \end{aligned} \tag{1.1}$$

Here x_{1002} and y_{1002} are constants which depend on the coefficients of the form H_2 and H_3 in the expansion (1.1) and

$$q_i = \sqrt{2r_i}\sin\phi_i \quad \text{and} \quad p_i = \sqrt{2r_i}\cos\phi_i \quad (i = 1, 2).$$

We may note that q_i's are the generalized co-ordinates and p_i's are the generalized momenta of the infinitesimal mass m_3. If $x_{1002}^2 + y_{1002}^2 \neq 0$, then equilibrium is unstable.

For $\omega_1 = 3\omega_2$.

For $\omega_1 = 3\omega_2$ the Hamiltonian can be reduced to the form

$$\begin{aligned} H &= \omega_2 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + \frac{1}{3} w_2 r_2 \sqrt{r_1 r_2} \\ &\quad \times \sqrt{3(x_{1003}^2 + y_{1003}^2)}\sin(\phi_1 + 3\phi_2) + o(r_1 + r_2)^{\frac{5}{2}}\,. \end{aligned} \tag{1.2}$$

The constants $c_{20}, c_{11}, c_{02}, x_{1003}$ and y_{1003} in (1.2) depend on the coefficients of the forms H_2, H_3 and H_4. The equilibrium position is unstable if the inequalities $x_{1003}^2 + y_{1003}^2 \neq 0$ and $3\omega_2\sqrt{x_{1003}^2 + y_{1003}^2} \geq |c_{20} + 3c_{11} + 9c_{02}|$ are fulfilled.

2(a) Equations of motion and location of L_4

We shall adopt the notation and terminology of Szebehely (1967) and Sharma Ravinder *et al.* (2001). As a consequence the distance between the primaries does not change and is taken equal to one; the sum of the masses of the primaries is also taken as one. The unit of time is chosen so as to make the gravitational constant unity. Using dimensionless variables, the equations of motion of the infinitesimal mass m_3 in the synodic co-ordinate system (x, y) are

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x\,, \\ \ddot{y} + 2n\dot{x} &= \Omega_y\,, \end{aligned} \qquad \text{[Sharma Ravinder et al. (2001)]} \tag{2.1}$$

where

$$
\begin{aligned}
\Omega &= \frac{n^2}{2}(m_1 r_1^2 + m_2 r_2^2) + \left(\frac{1}{r_1} + \frac{A_1}{2r_1^3} - \frac{P}{r_1}\right)m_1 \\
&\quad + \left(\frac{1}{r_2} + \frac{A_1'}{2r_2^3} + \frac{3A_2' y_3}{2r_2^2} - \frac{P'}{r_2}\right)m_2\,, \\
m_1 &= \text{mass of the bigger primary,} \\
m_2 &= \text{mass of the smaller primary,} \\
\mu &= \frac{m_2}{m_1+m_2} \le \frac{1}{2} \Rightarrow m_1 = 1-\mu\,, \\
r_1^2 &= (x-\mu)^2 + y^2\,, \\
r_2^2 &= (x - +1 - \mu)^2 + y^2\,, \\
P &= \frac{\text{Radiation pressure due to the bigger primary}}{\text{Gravitational force due to the bigger primary}}\,, \\
P' &= \frac{\text{Radiation pressure due to the smaller primary}}{\text{Gravitational force due to the smaller primary}}\,, \\
A_1 &= \frac{a^2 - c^2}{5R_2}, \quad A_1' = \frac{2a'^2 - c'^2 - b'^2}{5R_2}\,, \\
A_2' &= \frac{b'^2 - a'^2}{5R^2}, \quad 0 < A_1, A_1', A_2', P, P' \ll 1\,,
\end{aligned}
$$

a and c are the lengths of the semi-axes of the oblate body of mass m_1,
a', b' and c' are the lengths of the semi-axes of the triaxial body of mass m_2,
R =dimensional distance between the primaries.

The mean motion n of the primaries is given by

$$n = 1 + \frac{3}{4}A_1 + \frac{3}{4}A_1'\,.$$

It may be observed that n is independent of the parameters A_2', P and P'.

2(b) Location of the librations point L_4

The libration points are the solutions of the equations

$$\Omega_x = 0 \quad \text{and} \quad \Omega_y = 0$$

and the co-ordinates of L_4 are given by

$$
\begin{aligned}
x &= -\frac{1}{2} + \mu + \alpha_1 A_1 + \alpha_1' A_1' + \gamma_1' A_2' + \beta_1 P + \beta_1' P'\,, \\
y &= \frac{\sqrt{3}}{2} + \alpha_2 A_1 + \alpha_2' A_1' + \gamma_2' A_2' + \beta_2 P + \beta_2' P'\,,
\end{aligned}
$$

where

$$\alpha_1 = -\frac{1}{2}\,, \qquad \alpha'_1 = \frac{1}{2}\,,$$

$$\alpha_2 = -\frac{1}{2\sqrt{3}}\,, \qquad \alpha'_2 = -\frac{1}{2\sqrt{3}}\,,$$

$$\gamma'_1 = \frac{7}{8} + \frac{1}{2(1-2\mu)}\,, \qquad \gamma'_2 = \frac{\sqrt{3}}{2}\left(\frac{5}{4} - \frac{1}{3(1-\mu)}\right),$$

$$\beta_1 = \frac{1}{3}\,, \qquad \beta'_1 = \frac{1}{3}\,,$$

$$\beta_2 = -\frac{1}{3\sqrt{3}}\,, \qquad \beta'_2 = -\frac{1}{3\sqrt{3}}\,.$$

2(c) First order normalization

Now, we shall determine the normalized form of the Hamiltonian by following the procedure of Hallan *et al.* (2000).

The Lagrangian is given by

$$\begin{aligned} L &= \frac{1}{2}\{\dot{x}^2 + \dot{y}^2 + n^2(x^2+y^2) + 2n(x\dot{y} - y\dot{x})\} \\ &\quad + m_1(1-P)\left(\frac{1}{r_1} + \frac{A_1}{2r_1^3}\right) + m_2(1-P')\left(\frac{1}{r_2} + \frac{A'_1}{2r_2^3} + \frac{3A'_2 y^2}{2r_2^5}\right). \end{aligned}$$

Shifting the origin to $L_4(x,y)$, we have

$$\begin{aligned} L &= \frac{1}{2}\Bigg\{(\dot{x}^2+\dot{y}^2) + \left(x - \frac{\gamma}{2}\right)^2 + 2\left(x - \frac{\gamma}{2}\right) \\ &\quad \times (\alpha_1 A_1 + \alpha'_1 A'_1 + \gamma'_1 A'_2 + \beta_1 P + \beta'_1 P')\Bigg\} \\ &\quad + \frac{1}{2}\left\{\left(y + \frac{\sqrt{3}}{2}\right)^2 + 2\left(y + \frac{\sqrt{3}}{2}\right)(\alpha_2 A_1 + \alpha'_2 A'_1 + \gamma'_2 A'_2 + \beta_2 P + \beta'_2 P')\right\} \\ &\quad + \left(x - \frac{\gamma}{2} + \alpha_1 A_1 + \alpha'_1 A'_1 + \gamma'_1 A'_2 + \beta_1 P + \beta'_1 P'\right)\dot{y} \\ &\quad - \left(y + \frac{\sqrt{3}}{2} + \alpha_2 A_1 + \alpha'_2 A'_1 + \gamma'_2 A'_2 + \beta_2 P + \beta'_2 P'\right)\dot{x} \\ &\quad + \frac{3}{4}(A_1 + A'_1)\left\{\left(x - \frac{\gamma}{2}\right)^2 + \left(y + \frac{\sqrt{3}}{2}\right)^2 + \left(x - \frac{\gamma}{2}\right)\dot{y} - \left(y + \frac{\sqrt{3}}{2}\right)\dot{x}\right\} \\ &\quad + m_1\left(\frac{1}{r_1} + \frac{A_1}{2r_1^3} - \frac{P}{r_1}\right) + m_2\left(\frac{1}{r_2} + \frac{A'_1}{2r_2^3} + \frac{3A'_2}{2r_2^5}\left(y + \frac{\sqrt{3}}{2}\right)^2 - \frac{P'}{r_2}\right), \end{aligned}$$

where $\gamma = 1 - 2\mu$.
Expanding L in power series of x and y, we get

$$L = L_0 + L_1 + L_2 + L_3 + L_4 + \dots ,$$

where

$$\begin{aligned}
L_0 &= \frac{11+\gamma^2}{8} + \alpha_3 A_1 + \alpha_3' A_1' + \gamma_3' A_2' - \frac{1}{2}(1+\gamma)P - \frac{1}{2}(1-\gamma)P', \\
L_1 &= -\frac{\sqrt{3}}{24}\left(12 + 5A_1 + 5A_1' + \gamma_4' A_2' - \frac{8}{3}P - \frac{8}{3}P'\right)\dot{x} \\
&\quad -\frac{1}{8}\left\{4\gamma + \alpha_5 A_1 + \alpha_5' A_1' + \gamma_5' A_2' - \frac{8}{3}P + \frac{8}{3}P'\right\}\dot{y}, \\
L_2 &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}(4 + 3A_1 + 3A_1')(x\dot{y} - y\dot{x}) \\
&\quad + \frac{3}{16}\Big[2 + (5+4\gamma)A_1 + (5-4\gamma)A_1' + \gamma_6' A_2' \\
&\quad + \frac{2}{3}(1-3\gamma)P + \frac{2}{3}(1+3\gamma)P'\Big]x^2 \\
&\quad - \frac{\sqrt{3}}{8}\Big[6\gamma + (6+13\gamma)A_1 - (6-13\gamma)A_1' + \gamma_7' A_2' - \frac{2}{3}(3-\gamma)P \\
&\quad + \frac{2}{3}(3+\gamma)P'\Big]xy \\
&\quad + \frac{3}{16}\left[6 + 11A_1 + 11A_1' + \gamma_8' A_2' - \frac{2}{3}(1-3\gamma)P - \frac{2}{3}(3+\gamma)P'\right]y^2, \\
L_3 &= -\frac{1}{32}\Big\{14\gamma + (-6+25\gamma)A_1 + (6+25\gamma)A_1' + \gamma_9' A_2' + \frac{4}{3}(4+\gamma)P \\
&\quad + \frac{4}{3}(-4+\gamma)P'\Big\}x^3
\end{aligned}$$

$$\begin{aligned}
&-\frac{\sqrt{3}}{32}\Big\{6 + (43+60\gamma)A_1 + (43-60\gamma)A_1' + \gamma_{10}' A_2' \\
&-\frac{4}{3}(-8+21\gamma)P - \frac{4}{3}(-8-21\gamma)P'\Big\}x^2 y \\
&+\frac{3}{32}\Big\{22\gamma + (22+65\gamma)A_1 + (-22+65\gamma)A_1' + \gamma_{11}' A_2' \\
&+\frac{4}{3}(2+3\gamma)P + \frac{4}{3}(-2+3\gamma)P'\Big\}xy^2 \\
&-\frac{\sqrt{3}}{32}\left\{6 + 23A_1 + 23A_1' + \gamma_{12}' A_2' - \frac{4}{3}(2-9\gamma) - \frac{4}{3}(2+9\gamma)P'\right\}y^3,
\end{aligned}$$

$$
\begin{aligned}
L_4 \quad = \quad & -\frac{1}{256}\Big\{74+\alpha_{13}A_1+\alpha'_{13}A'_1+\gamma'_{13}A'_2-\frac{2}{3}(-87+113\gamma)P \\
& -\frac{2}{3}(-87-113\gamma)P'\Big\}x^4 \\
& +\frac{5\sqrt{3}}{192}\Big\{30\gamma+\alpha_{14}A_1+\alpha'_{14}A'_1+\gamma'_{14}A'_2+\frac{2}{3}(63-\gamma)P \\
& +\frac{2}{3}(-63-\gamma)P'\Big\}x^3y \\
& +\frac{3}{128}\Big\{82+\alpha_{15}A_1+\alpha'_{15}A'_1+\gamma'_{15}A'_2+\frac{2}{3}(111-169\gamma)P \\
& +\frac{2}{3}(111+169\gamma)P'\Big\}x^2y^2 \\
& -\frac{5\sqrt{3}}{64}\Big\{18\gamma+\alpha_{16}A_1+\alpha'_{16}A'_1+\gamma'_{16}A'_2+\frac{2}{3}(21+5\gamma)P \\
& +\frac{2}{3}(-21+5\gamma)P'\Big\}xy^3 \\
& +\frac{3}{256}\Big\{2+65A_1+65A'_1+\gamma'_{17}A'_2+\frac{2}{3}(-29+91\gamma)P \\
& +\frac{2}{3}(-29-91\gamma)P'\Big\}y^4 \,. \qquad (2.2)
\end{aligned}
$$

The Hamiltonian function is given by

$$
\begin{aligned}
H(x,y,p_x,p_y) \quad = \quad & \frac{1}{2}(p_x^2+p_y^2)+n(yp_x-xp_y)-\frac{m_1}{r_1}-\frac{m_2}{r_2}-\frac{m_1}{2r_1^3}A_1 \\
& -\frac{m_2}{2r_2^3}A'_1-\frac{3}{2}\frac{m_2}{r_2^5}y^2A'_2+\frac{m_1}{r_1}P+\frac{m_2}{r_2}P'.
\end{aligned}
$$

The translation given by

$$x \to x-\frac{\gamma}{2}+\alpha_1A_1+\alpha'_1A'_1+\gamma'_1A'_2+\beta_1P+\beta'_1P' \ ,$$

$$y \to y+\frac{\sqrt{3}}{2}+\alpha_2A_1+\alpha'_2A'_1+\gamma'_2A'_2+\beta_2P+\beta'_2P' \ ,$$

$$p_x \to p_x-n\bigg(\frac{\sqrt{3}}{2}+\alpha_2A_1+\alpha'_2A'_1+\gamma'_2A'_2+\beta_2P+\beta'_2P'\bigg) \ ,$$

$$p_y \to p_y+n\bigg(-\frac{\gamma}{2}+\alpha_1A_1+\alpha'_1A'_1+\gamma'_1A'_2+\beta_1P+\beta'_1P'\bigg) \ ,$$

transforms the Hamiltonian H to

$$\begin{aligned}
H &= \frac{1}{2}(p_x^2+p_y^2)+n(yp_x-xp_y) \\
&\quad -n^2\Big\{y\Big(\frac{\sqrt{3}}{2}+\alpha_2A_1+\alpha_2'A_1'+\gamma_2'A_2'+\beta_2P+\beta_2'P'\Big) \\
&\qquad +x\Big(-\frac{\gamma}{2}+\alpha_1A_1+\alpha_1'A_1'+\gamma_1'A_2'+\beta_1P+\beta_1'P'\Big)\Big\} \\
&\quad -\frac{n^2}{2}\Big\{\Big(\frac{\sqrt{3}}{2}+\alpha_2A_1+\alpha_2'A_1'+\gamma_2'A_2'+\beta_2P+\beta_2'P'\Big)^2 \\
&\qquad +\Big(-\frac{\gamma}{2}+\alpha_1A_1+\alpha_1'A_1'+\gamma_1'A_2'+\beta_1P+\beta_1'P'\Big)^2\Big\} \\
&\quad -\frac{m_1}{r_1}-\frac{m_2}{r_2}-\frac{m_1}{2r_1^3}A_1-\frac{m_2}{2r_2^3}A_1'-\frac{3m_2}{2r_2^5}\Big(y+\frac{\sqrt{3}}{2}\Big)^2A_2'+\frac{m_1}{r_1}P+\frac{m_2}{r_2}P'.
\end{aligned}$$

Substituting the expansions of $r_1^{-1}, r_2^{-1}, r_1^{-3}, r_2^{-3}, r_1^{-5}$ and r_2^{-5} in power series of x and y, we obtain $H=\sum_{k=0}^{\infty} H_k$, where H_k = the sum of the terms of kth degree homogenous in variables x, y, p_x, p_y.
Now

$$\begin{aligned}
H_0 &= -L_0, \\
H_1 &= 0, \\
H_2 &= \frac{1}{2}(p_x^2+p_y^2)+n(yp_x-xp_y)+Ex^2+Fy^2+2Gxy, \\
H_3 &= -L_3 \quad \text{and} \quad H_4=-L_4\,,
\end{aligned}$$

where

$$\begin{aligned}
E &= \frac{1}{16}\{2-3(1+4\gamma)A_1-3(1-4\gamma)A_1'\}+\gamma_{18}'A_2'-\frac{1}{8}(1-3\gamma)P \\
&\qquad -\frac{1}{8}(1+3\gamma)P', \\
F &= -\frac{1}{16}(10+21A_1+21A_1')+\gamma_{19}'A_2'+\frac{1}{8}(1-3\gamma)P+\frac{1}{8}(1+3\gamma)P', \\
G &= \frac{\sqrt{3}}{8}\{6\gamma+\alpha_{20}A_1+\alpha_{20}'A_1'\}+\gamma_{20}'A_2'+\frac{2}{3}(-3+\gamma)P+\frac{2}{3}(3+\gamma)P'.
\end{aligned}$$

To investigate the stability of motion as in Whittaker (1965), we consider the following set of linear equations in the variables x and y:

$$\begin{aligned} -\lambda p_x &= \frac{\partial H_2}{\partial x} = 2Ex + Gy - np_y, \\ -\lambda p_y &= \frac{\partial H_2}{\partial y} = 2Fy + Gx + np_x, \\ \lambda x &= \frac{\partial H_2}{\partial p_x} = p_x + ny, \\ \lambda y &= \frac{\partial H_2}{\partial p_y} = p_y - nx, \end{aligned} \tag{2.3}$$

i.e. $AX = 0$,
where

$$A = \begin{pmatrix} 2E & G & \lambda - n & \\ G & 2F & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}.$$

The Equations (2.3) will have a non zero solution if and only if $\det(A) = 0$, which implies that

$$\lambda^4 + 2\lambda^2(E + F + n^2) + EF - G^2 - 2n^2(E + F) + n^4 = 0,$$

or

$$\begin{aligned} 16\lambda^4 + \lambda^2\{8(2 - 3\gamma A_1 + 3\gamma A_1') - 48(1-\gamma)A_2'\} + 9(1-\gamma^2)(3 + 13A_1 + 13A_1') \\ + \frac{45}{4}(7 - 4\gamma - 3\gamma^2)A_2' + 6(1-\gamma^2)(P + P') = 0\,. \end{aligned} \tag{2.4}$$

The characteristic Equation (2.4) is quadratic in λ^2 whose discriminant is given by

$$\begin{aligned} \text{Disc} \;=\; & \{8(2 - 3\gamma A_1 + 3\gamma A_1') - 48(1-\gamma)A_2'\}^2 - 64\{9(1-\gamma^2) \\ & \times (3 + 13A_1 + 13A_1') + \frac{45}{4}(7 - 4\gamma - 3\gamma^2)A_2' - 6(1-\gamma^2)(P + P')\}. \end{aligned} \tag{2.5}$$

Dics $= 0$, if

$$\begin{aligned} &\gamma^2(27 + 117A_1 + 117A_1' + \frac{135}{4}A_2' + 6P + 6P') + \gamma(-12A_1 + 12A_1' + 69A_2') \\ &+(-23 - 117A_1 - 117A_1' - \frac{411}{4}A_2' - 6P - 6P') = 0 \end{aligned}$$

or

$$\begin{aligned} \gamma \;=\; & 0.9229582\ldots + 0.5700037\ldots A_1 + 0.1255592\ldots A_1' \\ & +0.2069804\ldots A_2' + 0.0178348\ldots (P + P') \end{aligned}$$

As $\gamma = 1 - 2\mu$,

$$\begin{aligned}\mu &= \mu_o - 0.285002A_1 - 0.06278A_1' - 0.10349A_2' - 0.00891747(P+P') \\ &\equiv \mu_{co} \quad \text{(say)},\end{aligned}$$

where $\mu_o = 0.0385208965\ldots$.
Stability is assured only when discriminant of the Equation (2.4) is greater than zero, implies that

$$\mu < \mu_{co}.$$

When the discriminant of the Equation (2.4) is positive, let its roots be $\pm i\omega_1$ and $\pm i\omega_2$, ω_1 and ω_2 being the long and short periodic frequencies and are related to each other as

$$\omega_1^2 + \omega_2^2 = 1 - \frac{3}{2}\gamma A_1 + \frac{3}{2}\gamma A_1' - 3(1-\gamma)A_2', \tag{2.6a}$$

$$\begin{aligned}\omega_1^2\omega_2^2 &= \frac{9}{16}\{(1-\gamma^2)(3+13A_1+13A_1') + \frac{5}{4}(7-4\gamma-3\gamma^2)A_2' \\ &\quad + \frac{2}{3}(1-\gamma^2)(P+P')\}\end{aligned} \tag{2.6b}$$

$(0 < \omega_2 < \frac{1}{\sqrt{2}} < \omega_1 < 1)$.
For the resonance case $\omega_1 = 2\omega_2$ using (2.6a) and (2.6b) the value of μ_{c1} is

$$\begin{aligned}\mu_{c1} &= 0.0242939 - 0.17907278A_1 - 0.36851A_1' - 0.05968A_2' \\ &\quad -0.005536495(P+P')\end{aligned}$$

and for the resonance case $\omega_1 = 3\omega_2$ the value of μ_{c2} is

$$\begin{aligned}\mu_{c2} &= 0.013516 - 0.09938302A_1 - 0.01938A_1' - 0.03093A_2' \\ &\quad -0.003045283(P+P').\end{aligned}$$

3. Determination of the normal co-ordinates

We follow the method of Whittaker (1965) to determine the normal co-ordinates. Applying the transformation $(x, y, p_x, p_y) \to (q_1', q_2', p_1', p_2')$ given by

$$X = JT,$$

where

$$X = \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}, \qquad T = \begin{pmatrix} q_1' \\ q_2' \\ p_1' \\ p_2' \end{pmatrix},$$

J is the square matrix given by

$$J = (J_{ij}).$$

And J_{ij} are given in Appendix A, obtained by the procedure adopted by Sanjay *et al* (2000).

The Hamiltonian H reduces to

$$H = \frac{1}{2}(p_1^{'2} - p_2^{'2} + \omega_1^2 q_1^{'2} - \omega_2^2 q_2^{'2}) + \sum_{\substack{\alpha+\beta=3 \\ \alpha=\alpha_1+\alpha_2 \\ \beta=\beta_1+\beta_2}}^{\infty} h_{\alpha_1\alpha_2\beta_1\beta_2} q_1^{'\alpha_1} q_2^{'\alpha_2} p_1^{'\beta_1} p_2^{'\beta_2}\,. \quad (3.1)$$

Here coefficients $h_{\alpha_1\alpha_2\beta_1\beta_2}$ upto order four are given in Appendix B.

We shall now perform the following complex canonical transformation

$$(q_1', q_2', p_1', p_2') \to (q_1'', q_2'', p_1'', p_2'')$$

i.e.

$$\begin{aligned} q_1' &= \frac{1}{2}q_1'' + \frac{1}{\omega_1} i p_1'', \qquad p_1' = \frac{1}{2} i\omega_1 q_1'' + p_1''\,, \\ q_2' &= -\frac{1}{2}q_2'' + \frac{1}{\omega_2} i p_2'', \qquad p_2' = -\frac{1}{2} i\omega_2 q_2'' + p_2''\,. \end{aligned}$$

The Hamiltonian H given by (3.1) reduces to

$$H = i\omega_1 q_1'' p_1'' + i\omega_2 q_2'' p_2'' + \sum_{\substack{\alpha+\beta=3 \\ \alpha=\alpha_1+\alpha_2 \\ \beta=\beta_1+\beta_2}}^{\infty} h'_{\alpha_1\alpha_2\beta_1\beta_2} q_1^{''\alpha_1} q_2^{''\alpha_2} p_1^{''\beta_1} p_2^{''\beta_2}, \quad (3.2)$$

where

$$h'_{\alpha_1\alpha_2\beta_1\beta_2} = x_{\alpha_1\alpha_2\beta_1\beta_2} + i y_{\alpha_1\alpha_2\beta_1\beta_2}$$

$x_{\alpha_1\alpha_2\beta_1\beta_2}$ and $y_{\alpha_1\alpha_2\beta_1\beta_2}$ (of order three) in terms of $h'_{\alpha_1\alpha_2\beta_1\beta_2}$ are given in Appendix C.

Now, we shall use the Birkhoff's transformation (1927) $(q_1'', q_2'', p_1'', p_2'') \to (q_1''', q_2''', p_1''', p_2''')$ defined by the generating function

$$S = q_1'' p_1''' + q_2'' p_2''' + S_3 + S_4,$$

where

$$q_i''' = \frac{\partial S}{\partial p_i'''} = q_i'' + \frac{\partial S_3}{\partial p_i'''} + \frac{\partial S_4}{\partial p_i'''}, \qquad (i = 1, 2). \qquad \text{(Szebehely 1967)}$$

Putting these values in (3.2) and letting the new Hamiltonian to H', we shall have

$$\begin{aligned} H(q_1'', q_2'', p_1'', p_2'') &= H\left(q_1'', q_2'', p_1''' + \frac{\partial S_3}{\partial q_1''} + \frac{\partial S_4}{\partial q_1''}, p_2''' + \frac{\partial S_3}{\partial q_2''} + \frac{\partial S_4}{\partial q_2''}\right) + \frac{\partial S_3}{\partial t} + \frac{\partial S_4}{\partial t} \\ &= H'(q_1''', q_2''', p_1''', p_2''') \\ &= H'\left(q_1'' + \frac{\partial S_3}{\partial p_1'''} + \frac{\partial S_4}{\partial p_1'''}, q_2''' + \frac{\partial S_3}{\partial p_2'''} + \frac{\partial S_4}{\partial p_2'''}, p_1''', p_2'''\right). \end{aligned}$$

Expanding by Taylor's and equating the terms of same degree on both sides, we get

$$H_2'(q_1'', q_2'', p_1''', p_2''') = H_2(q_1'', q_2'', p_1''', p_2'''), \tag{3.3a}$$

$$\sum_{i=1}^{2}\left(-\frac{\partial S_3}{\partial p_i'''}\frac{\partial H_2'}{\partial q_i''} + \frac{\partial S_3}{\partial q_i''}\frac{\partial H_2'}{\partial p_i'''}\right) + H_3(q_1'', q_2'', p_1''', p_2''') = H_3'(q_1'', q_2'', p_1''', p_2'''), \tag{3.3b}$$

$$\sum_{i=1}^{2}\left(-\frac{\partial S_4}{\partial p_i'''}\frac{\partial H_2'}{\partial q_i''} + \frac{\partial S_4}{\partial q_i''}\frac{\partial H_2}{\partial p_i'''}\right) + K_4$$

$$= K_4 - H_4 + H_4' + \sum_{i=1}^{2}\left(-\frac{\partial S_3}{\partial q_i''}\frac{\partial H_3}{\partial p_i'''} + \frac{\partial S_3}{\partial p_i'''}\frac{\partial H_3'}{\partial q_i''}\right), \tag{3.3c}$$

where $K_4 =$ The terms other than the homogeneous in $q_1 p_1$ and $q_2 p_2$.
Since the variations q_i and p_i are small, so by implicit function theorem we may use q_i''' and p_i''' in place of q_i'' and p_i'' in (3.3). Also $\frac{\partial S_3}{\partial t} = 0$ and $\frac{\partial S_4}{\partial t} = 0$ since our system is autonomous.

3(a) Hamiltonian H_3 in the resonance case $\omega_1 = 2\omega_2$

Now by taking

$$H_3 = \sum_{\substack{\alpha+\beta=3 \\ \alpha=\alpha_1+\alpha_2 \\ \beta=\beta_1+\beta_2}} h'_{\alpha_1\alpha_2\beta_1\beta_2} q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2}, \tag{3.4}$$

$$S_3 = \sum_{\substack{\alpha+\beta=3 \\ \alpha=\alpha_1+\alpha_2 \\ \beta=\beta_1+\beta_2}} g_{\alpha_1\alpha_2\beta_1\beta_2} q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2}, \tag{3.5}$$

and putting the values of H_2 and H_2', S_3 and H_3 from (3.2), (3.3a), (3.4) and (3.5) in (3.3b) we have

$$H_3' = \sum_{\substack{\alpha+\beta=3 \\ \alpha=\alpha_1+\alpha_2 \\ \beta=\beta_1+\beta_2}} [i g_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1) + \omega_2(\alpha_2-\beta_2)\} + h'_{\alpha_1\alpha_2\beta_1\beta_2}]$$

$$\times\, q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2}$$

or

$$H'_3 = \sum_{\substack{\alpha+\beta=3\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2\\ (\alpha_1,\alpha_2,\beta_1,\beta_2)\in R_1}} [ig_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)\}+h'_{\alpha_1\alpha_2\beta_1\beta_2}] \times q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2}$$
$$+ \sum_{\substack{\alpha+\beta=3\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2\\ (\alpha_1,\alpha_2,\beta_1,\beta_2)\in NR_1}} [ig_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)\}+h'_{\alpha_1\alpha_2\beta_1\beta_2}] \times q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2}, \quad (3.6)$$

where R_1 = set of combination of α_i's and β_i's corresponding to the resonance case $\omega_1 = 2\omega_2$
i.e.

$$R_1 = \{(\alpha_1 = 1, \alpha_2 = 0, \beta_1 = 0, \beta_2 = 2)\,, \quad (\alpha_1 = 0, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 0)\}$$

and NR_1 = set of combination of α_i's and β_i's corresponding to the non resonance case

$$NR_1 = \{(\alpha_1,\alpha_2,\beta_1,\beta_2) | \alpha_1+\alpha_2+\beta_1+\beta_2 = 3, (\alpha_1,\alpha_2,\beta_1,\beta_2) \notin R_1\}.$$

We choose values of $g_{\alpha_1\alpha_2\beta_1\beta_2}$ in such a way that all the terms of second $\sum$ of R.H.S of (3.6) are zero. Thus, we have

$$ig_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)\}+h'_{\alpha_1\alpha_2\beta_1\beta_2} = 0$$

or

$$g_{\alpha_1\alpha_2\beta_1\beta_2} = \frac{ih'_{\alpha_1\alpha_2\beta_1\beta_2}}{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)} \forall \alpha_1,\alpha_2,\beta_1,\beta_2$$
such that $\alpha_1+\alpha_2+\beta_1+\beta_2 = 3$ and $(\alpha_1,\alpha_2,\beta_1,\beta_2) \notin R_1$.

Substituting the values of $g_{\alpha_1\alpha_2\beta_1\beta_2}$ in (3.6), we have

$$H'_3 = h'_{1002} q_1''' p_2'''^2 + h'_{0120} q_2''' p_1'''^2, \quad (3.7)$$

where

$$h'_{1002} = x_{1002} + iy_{1002}, \qquad h'_{0210} = (y_{1002} + ix_{1002})\left(-\frac{\omega_1}{2}\right)^{-1}\left(\frac{\omega_2}{2}\right)^2.$$

Note. $(\alpha_1-\beta_1)\omega_1 + (\alpha_2-\beta_2)\omega_2 = 0$ when $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in R_1$.

3(b) Hamiltonian H_3 in the resonance case $\omega_1 = 3\omega_2$

Now by taking

$$H_4 = \sum_{\substack{\alpha+\beta=4\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2}} h'_{\alpha_1\alpha_2\beta_1\beta_2} q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2}$$

and

$$S_4 = \sum_{\substack{\alpha+\beta=4\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2}} g_{\alpha_1\alpha_2\beta_1\beta_2} q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2},$$

the L.H.S. of (3.3)

$$\begin{aligned}
&= \sum_{i=1}^{2}\left(-\frac{\partial S_4}{\partial p_i'''}\frac{\partial H_2'}{\partial q_i''} + \frac{\partial S_4}{\partial q_i''}\frac{\partial H_2}{\partial p_i'''}\right) + K_4 \\
&= \sum_{\substack{\alpha+\beta=4\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2}} \{ig_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)\}q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2} \\
&\quad + \sum_{\substack{\alpha+\beta=4\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2\\ (\alpha_1,\alpha_2,\beta_1,\beta_2)\notin HT}} h'_{\alpha_1\alpha_2\beta_1\beta_2} q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2} \\
&= \sum_{\substack{\alpha+\beta=4\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2\\ (\alpha_1,\alpha_2,\beta_1,\beta_2)\notin HT}} [ig_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)\} + h'_{\alpha_1\alpha_2\beta_1\beta_2}] \\
&\qquad \times q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\beta_1} p_2'''^{\beta_2},
\end{aligned}$$

where

$HT =$ Set of α_i's and β_i's corresponding to the terms homogeneous in q_1p_1 and q_2p_2

i.e.

$$\begin{aligned}
HT &= \{(\alpha_1=2,\alpha_2=0,\beta_1=2,\beta_2=0),\{(\alpha_1=1,\alpha_2=1,\beta_1=1,\beta_2=1), \\
&\quad (\alpha_1=0,\alpha_2=2,\beta_1=0,\beta_2=2)\}
\end{aligned}$$

and we note that $(\alpha_1-\beta_1)\omega_1 + (\alpha_2-\beta_2)\omega_2 = 0$ when $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in HT$.

The L.H.S. of (3.3) becomes

$$\sum_{\substack{\alpha+\beta=4\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2\\ (\alpha_1,\alpha_2,\beta_1,\beta_2)\in NR_2}} [ig_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)\}+h'_{\alpha_1\alpha_2\beta_1\beta_2}]q_1'''^{\alpha_1}q_2'''^{\alpha_2}p_1'''^{\beta_1}p_2'''^{\beta_2}$$

$$+\sum_{\substack{\alpha+\beta=3\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2\\ (\alpha_1,\alpha_2,\beta_1,\beta_2)\in R_2}} [ig_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)\}+h'_{\alpha_1\alpha_2\beta_1\beta_2}]q_1'''^{\alpha_1}q_2'''^{\alpha_2}p_1'''^{\beta_1}p_2'''^{\beta_2}$$

$$=\sum_{\substack{\alpha+\beta=4\\ \alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2\\ (\alpha_1,\alpha_2,\beta_1,\beta_2)\in NR_2}} [ig_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)\}+h_{\alpha_1\alpha_2\beta_1\beta_2}]$$

$$\times q_1'''^{\alpha_1}q_2'''^{\alpha_2}p_1'''^{\beta_1}p_2'''^{\beta_2}+h'_{1003}q_1'''p_2'''^3+h'_{0310}q_2'''^3p_1''', \tag{3.8}$$

where

R_2 = set of combination of α_i's and β_i's corresponding to the resonance case $\omega_1=3\omega_2$.

i.e.

$$R_2=\{(\alpha_1=1,\alpha_2=0,\beta_1=0,\beta_2=3),(\alpha_1=0,\alpha_2=3,\beta_1=1,\beta_2=0)\}$$

and

NR_2 = set of combination of α_i's and β_i's corresponding to the non resonance case

i.e

$$NR_2=\{(\alpha_1,\alpha_2,\beta_1,\beta_2)|\alpha_1+\alpha_2+\beta_1+\beta_2=3,(\alpha_1,\alpha_2,\beta_1,\beta_2)\notin R_2,HT\}.$$

Note. $(\alpha_1-\beta_1)\omega_1+(\alpha_2-\beta_2)\omega_2=0$ when $(\alpha_1,\beta_1,\alpha_2,\beta_2)\in R_2$.

We choose values of $g_{\alpha_1\alpha_2\beta_1\beta_2}$ such that all the coefficients in (3.8) are zero. Thus,

$$ig_{\alpha_1\alpha_2\beta_1\beta_2}\{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)\}+h'_{\alpha_1\alpha_2\beta_1\beta_2}=0$$

or

$$g_{\alpha_1\alpha_2\beta_1\beta_2}=\frac{ih'_{\alpha_1\alpha_2\beta_1\beta_2}}{\omega_1(\alpha_1-\beta_1)+\omega_2(\alpha_2-\beta_2)}\ \forall\ \alpha_1,\alpha_2,\beta_1,\beta_2$$

such that $\alpha_1+\alpha_2+\beta_1+\beta_2=4$ and $(\alpha_1,\alpha_2,\beta_1,\beta_2)\notin R_2,HT$.

Substituting the value of $g_{\alpha_1\alpha_2\beta_1\beta_2}$ in (3.8) , the L.H.S of (3.3) becomes

$$h'_{1003} q_1''' p_2'''^3 + h'_{0310} q_2'''^3 p_1''' \,. \tag{3.9}$$

Simplifying R.H.S. of (3.3) and using (3.9) we have

$$H'_4 = D + Q, \tag{3.10}$$

Where

$$\begin{aligned}
D &= -c_{20} q_1'''^2 p_1'''^2 + c_{11} q_1''' p_1''' q_2''' p_2''' - c_{02} q_2'''^2 p_2'''^2,\\
Q &= l_{1003} q_1''' p_2'''^3 + l_{0310} p_1''' q_2'''^3,\\
c_{20} &= -h'_{2020} - \frac{3\omega_1^2}{8}(x_{0030}^2 + y_{0030}^2) - \frac{3}{2}(x_{1020}^2 + y_{1020}^2) + \frac{1}{2}(x_{1011}^2 + y_{1011}^2)\\
&\quad -\frac{\omega_1^2}{2\omega_2(2\omega_1 - \omega_2)}(x_{0120}^2 + y_{0120}^2) + \frac{\omega_1^2\omega_2}{8(2\omega_1 + \omega_2)}(x_{0021}^2 + y_{0021}^2),\\
c_{11} &= h'_{1111} - \frac{2\omega_1^2}{\omega_1(\omega_1 - 2\omega_2)}(x_{1002}^2 + y_{1002}^2) + \frac{\omega_2^2\omega_1}{2(2\omega_2 + \omega_1)}(x_{0012}^2 + y_{0012}^2)\\
&\quad -\frac{\omega_1^2\omega_2}{2(2\omega_1 + \omega_2)}(x_{0021}^2 + y_{0021}^2) - \frac{2\omega_1^2}{\omega_2(2\omega_1 - \omega_2)}(x_{0120}^2 + y_{0120}^2)\\
&\quad +2(x_{0111}x_{1020} + y_{0111}y_{1020}) - \frac{4}{\omega_2}(x_{0201}x_{1011} + y_{1011}y_{0201}),\\
c_{02} &= h'_{0202} - \frac{3\omega_2^2}{8}(x_{0003}^2 + y_{0003}^2) + \frac{6}{\omega_2^2}(x_{0201}^2 + y_{0201}^2) - \frac{1}{2}(x_{0111}^2 + y_{0111}^2)\\
&\quad -\frac{\omega_2^2}{2\omega_1(\omega_1 - 2\omega_2)}(x_{1002}^2 + y_{1002}^2) - \frac{\omega_2^2\omega_1}{8(\omega_1 + 2\omega_2)}(x_{0012}^2 + y_{0012}^2),\\
h'_{2020} &= -\frac{3}{2}\omega_1^2 h_{0040} - \frac{3}{2\omega_1^2}h_{4000} - \frac{1}{2}h_{2020},\\
h'_{1111} &= \omega_1\omega_2 h_{0022} - \frac{1}{\omega_1\omega_2}h_{0220} + \frac{\omega_1}{\omega_2}h_{0220} + \frac{\omega_2}{\omega_1}h_{2002},\\
h'_{0202} &= -\frac{3}{2}\omega_2^2 h_{0004} - \frac{3}{2\omega_2^2}h_{0400} + \frac{1}{2}h_{0202},\\
l_{1003} &= x_{1003} + i y_{1003},
\end{aligned}$$

$$
\begin{aligned}
l_{0310} &= (x_{1003} - iy_{1003})\left(\frac{-\omega_1^2}{12}\right), \\
x_{1003} &= u_{1003} - \frac{9}{5}(x_{0120}x_{0012} + y_{0120}x_{0012}) - \frac{1}{\omega_2}(x_{1002}y_{1011} + x_{1011}y_{1002}) \\
&\quad + \frac{4}{\omega_2^2}(x_{1002}x_{0201} + y_{1002}y_{0201}) + \frac{3}{2}(x_{0003}x_{0111} + y_{0003}y_{0111}), \\
y_{1003} &= v_{1003} - \frac{9}{5}(x_{0120}y_{0012} - y_{0120}x_{0012}) - \frac{1}{\omega_2}(y_{1002}y_{1011} - x_{1011}x_{1002}) \\
&\quad + \frac{4}{\omega_2^2}(y_{1002}x_{0201} - x_{1002}y_{0201}) + \frac{3}{2}(y_{0003}x_{0111} - x_{0003}y_{0111}),
\end{aligned}
$$

$$
\begin{aligned}
u_{1003} &= \frac{1}{2}\omega_1 h_{0013} + \frac{1}{2\omega_2^3}h_{1300} - \frac{1}{2\omega_2}h_{1102} - \frac{\omega_1}{2\omega_2^2}h_{0211}, \\
v_{1003} &= -\frac{\omega_1}{2\omega_2}h_{0112} - \frac{1}{2}h_{1003} + \frac{1}{2\omega_2^2}h_{1201} + \frac{\omega_1}{2\omega_2^3}h_{0310}\,.
\end{aligned}
$$

4(a) Stability in the resonance case $\omega_1 = 2\omega_2$

In the last section we have seen that the normalized form of the Hamiltonian in this case can be written as

$$
H = i\omega_1 q_1''' p_1''' + i\omega_2 q_2''' p_2''' + h'_{1002} q_1''' p_2'''^2 + h'_{0210} q_2'''^2 p_1'''. \tag{4.1}
$$

We shall now apply Markeev's theorem(1978). For this we shall perform the following canonical transformation

$$
q_1''' = \frac{1}{\sqrt{\omega_1}}(\tilde{q}_1 - i\tilde{p}_1), \qquad q_2''' = \frac{1}{\sqrt{\omega_2}}(i\tilde{q}_2 - \tilde{p}_2) \tag{4.2}
$$

and then by another transformation

$$
\tilde{q}_j = \sqrt{2r_j}\sin(\phi_j - \theta_j), \quad \tilde{p}_j = \sqrt{2r_j}\cos(\phi_j - \theta_j), \quad (j = 1, 2), \tag{4.3}
$$

where $\theta_2 = 0$ and θ_1 is given by the relations

$$
\sin\theta_1 = \frac{y_{1002}}{\sqrt{x_{1002}^2 + y_{1002}^2}}, \qquad \cos\theta_1 = \frac{x_{1002}}{\sqrt{x_{1002}^2 + y_{1002}^2}}.
$$

The Hamiltonian (4.1) reduces to

$$
H = 2\omega_2 r_1 - \omega_2 r_2 - r_2\sqrt{r_1}\sqrt{(x^2 1002 + y_{1002}^2)\omega_2}\sin(\phi_1 + 2\phi_2) + o((r_1 + r_2)^2)
$$

It is known by Markeev's theorem that if

$$
x_{1002}^2 + y_{1002}^2 \neq 0,
$$

then the libration point will be unstable and if

$$x_{1002}^2 + y_{1002}^2 = 0 \qquad \text{and} \quad c_{20} + 2c_{11} + 4c_{20} \neq 0,$$

then the libration point will be Liapunov stable.
For the resonance case $\omega_1 = 2\omega_2$, we have

$$\begin{aligned} x_{1002}^2 + y_{1002}^2 \quad &= \quad 4.10802468 + 12.756443787A_1 + 19.4955166A_1' \\ &\quad -139.44377A_2' + \; 2.24793085P + 140.2725166P' \,. \end{aligned}$$

We have found that for no values of A_1, A_1', A_2', P and P',

$$x_{1002}^2 + y_{1002}^2 = 0.$$

Thus the libration point L_4 is always unstable.

Note. For the classical case when $A_1 = A_1' = A_2' = P = P' = 0$, we obtain

$$x_{1002}^2 + y_{1002}^2 = 4.10802468 \neq 0,$$

implies L_4 is unstable which agrees with Markeev's result.

4(b) Stability in the resonance case $\omega_1 = 3\omega_2$

In section 3(b) we have seen that the normalized form of the Hamiltonian in this case can be written as:

$$\begin{aligned} H \quad = \quad & i\omega_1 q_1''' p_1''' + i\omega_2 q_2''' p_2''' - c_{20} q_1'''^2 p_1'''^2 + c_{11} q_1''' p_1''' q_2''' p_2''' - c_{02} q_2'''^2 p_2'''^2 \\ & + l_{1003} q_1''' p_2'''^3 + l_{0310} q_2'''^3 p_1''' \,. \end{aligned}$$

If

$$x_{1003}^2 + y_{1003}^2 \neq 0,$$

then performing the transformation (4.2) and (4.3) (taking x_{1003} and y_{1003} in place of x_{1002} and y_{1002} respectively), the Hamiltonian (4.4) reduces to the form

$$\begin{aligned} H \quad = \quad & 3\omega_2 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 \\ & + \frac{1}{3}\omega_2 r_2 \sqrt{r_1 r_2} \sqrt{3(x_{1003}^2 + y_{1003}^2)} \sin(\phi_1 + 3\phi_2) + o(r_1 + r_2)^{\frac{5}{2}} . \end{aligned}$$

If we write

$$a = |c_{20} + 3c_{11} + 9c_{02}]| \quad \text{and} \quad d = |3\omega_2 \sqrt{x_{1003}^2 + y_{1003}^2}]|$$

then by Markeev's theorem, the libration point will be unstable if $a < d$ and will be stable if $a > d$. When $d = 0$, the motion will be obviously stable.

For the resonance case $\omega_1 = 3\omega_2$, we have

$$\begin{aligned} a &= |-4.17054 - 37.23732A_1 - 51.424659A_1' + 5654.354078A_2' \\ &\quad +1227.28554P + 4196.2211P'|, \\ d &= |23.2826 + 87.7002A_1 + 114.604124A_1' - 2068.50633A_2' \\ &\quad -318.84952P + 446.8862P'|\,. \end{aligned}$$

We have observed that for different values of A, A_1', A_2', P and P', we may have $a < d$ or $a > d$. For example

(i) when $A_1 = 0.0001$, $A_1' = 0.0001$, $A_2' = 0.0001$, $P = 0.0001$ and $P' = 0.0001$, $a = 3.0742619$, $d = 23.11036$, (i.e. $a < d$).

(ii) when $A_1 = 0.0001$, $A_1' = 0.0001$, $A_2' = 0.004$, $P = 0.0001$ and $P' = 0.0001$, $a = 18.28934$, $d = 15.29533$. (i.e. $a > d$)

Thus we see that the libration point L_4 is unstable in case (i) and stable in case (ii). Therefore libration point L_4 will be stable or unstable depending upon the values of A_1, A_1', A_2', P and P'. For the classical case, when $A_1 = A_1' = A_2' = P = P' = 0$, we obtain $a = 4.17054$ and $d = 23.2826$ implying L_4 is unstable which agrees with Markeev's result.

5. Conclusion

We have studied the non-linear stability of L_4 in the restricted three body problem with the resonance cases $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$, when the bigger primary is an oblate body and the smaller a triaxial body and both are sources of radiation. We have found that the libration point L_4 is always unstable in the resonance case $\omega_1 = 2\omega_2$ and in the case of $\omega_1 = 3\omega_2$, L_4 will be stable or unstable depending upon the values of the parameters A_1, A_1', A_2', P and P'.

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Appendix A

$$
\begin{aligned}
j_{11} &= j_{12} = 0\,, \\
j_{13} &= \frac{l_1}{2\omega_1 k_1}\{1+\alpha_{21}A_1+\alpha'_{21}A'_1+\gamma'_{21}A'_2+\beta_{21}P+\beta'_{21}P'\}, \\
j_{14} &= \frac{l_2}{2\omega_2 k_2}\{1+\alpha_{22}A_1+\alpha'_{22}A'_1+\gamma'_{22}A'_2+\beta_{22}P+\beta'_{22}P'\}, \\
j_{21} &= \frac{-4\omega_1}{l_1 k_1}\{1+\alpha_{23}A_1+\alpha'_{23}A'_1+\gamma'_{23}A'_2+\beta_{23}P+\beta'_{23}P'\}, \\
j_{22} &= \frac{4\omega_2}{l_2 k_2}\{1+\alpha_{24}A_1+\alpha'_{24}A'_1+\gamma'_{24}A'_2+\beta_{24}P+\beta'_{24}P'\}, \\
j_{23} &= \frac{3\sqrt{3}\gamma}{2\omega_1 l_1 k_1}\{1+\alpha_{25}A_1+\alpha'_{25}A'_1+\gamma'_{25}A'_2+\beta_{25}P+\beta_{25}P'\}, \\
j_{24} &= \frac{3\sqrt{3}\gamma}{2\omega_2 l_2 k_2}\{1+\alpha_{26}A_1+\alpha'_{26}A'_1+\gamma'_{26}A'_2+\beta_{26}P+\beta'_{26}P'\}, \\
j_{31} &= \frac{-\omega_1 m_1}{2 l_1 k_1}\left\{1+\alpha_{27}A_1+\alpha'_{27}A'_1+\left(\gamma'_{23}-\frac{8\gamma'_{19}}{m_1}\right)A'_2+\left(\beta_{23}-\frac{8\beta_{19}}{m_1}\right)P \right. \\
&\quad \left. +\left(\beta'_{23}-\frac{8\beta'_{19}}{m_1}\right)P'\right\}, \\
j_{32} &= \frac{\omega_2 m_2}{2 l_2 k_2}\left\{1+\alpha_{28}A_1+\alpha'_{28}A'_1+\left(\gamma'_{24}-\frac{8\gamma'_{19}}{m_1}\right)A'_2+\left(\beta_{24}-\frac{8\beta_{19}}{m_1}\right)P \right. \\
&\quad \left. +\left(\beta'_{24}-\frac{8\beta'_{19}}{m_1}\right)P'\right\}, \\
j_{33} &= -\left(1+\frac{3}{4}A_1+\frac{3}{4}A'_1\right)j_{23}, \\
j_{34} &= -\left(1+\frac{3}{4}A_1+\frac{3}{4}A'_1\right)j_{24}, \\
j_{41} &= -\omega_1^2 j_{23}, \\
j_{42} &= \omega_2^2 j_{24}, \\
j_{43} &= \frac{n_1}{2\omega_1 l_1 k_1}\left\{1+\alpha_{29}A_1+\alpha'_{29}A'_1+\left(\gamma'_{23}-\frac{8\gamma'_{19}}{n_1}\right)A'_2+\left(\beta_{23}-\frac{8\beta_{19}}{n_1}\right)P \right. \\
&\quad \left. +\left(\beta'_{23}-\frac{8\beta'_{19}}{n_1}\right)P'\right\}, \\
j_{44} &= \frac{n_2}{2\omega_2 l_2 k_2}\left\{1+\alpha_{30}A_1+\alpha'_{30}A'_1+\left(\gamma'_{24}-\frac{8\gamma'_{19}}{n_1}\right)A'_2+\left(\beta_{24}-\frac{8\beta_{19}}{n_1}\right)P \right. \\
&\quad \left. +\left(\beta'_{24}-\frac{8\beta'_{19}}{n_1}\right)P'\right\}.
\end{aligned}
$$

where

$$l_1 = \sqrt{9+\omega_1^2}, \qquad l_2 = \sqrt{9+\omega_2^2},$$

$$n_1 = 9-4\omega_1^2, \qquad n_2 = 9-4\omega_2^2,$$

$$m_1 = 1+4\omega_1^2, \qquad m_2 = 1+4\omega_2^2,$$

$$k_1 = \sqrt{2\omega_1^2-1}, \qquad k_2 = \sqrt{2\omega_2^2-1},$$

$$\alpha_3 = \frac{1}{16}(13+4\gamma+3\gamma^2), \qquad \alpha_3' = \frac{1}{16}(13-4\gamma+3\gamma^2),$$

$$\alpha_5 = (4+3\gamma), \qquad \alpha_5' = (-4+3\gamma)A_1',$$

$$\alpha_{13} = (285+200\gamma), \qquad \alpha_{13}' = (285-200\gamma),$$

$$\alpha_{14} = (-54+53\gamma), \qquad \alpha_{14}' = (54+53\gamma),$$

$$\alpha_{15} = (405+340\gamma), \qquad \alpha_{15}' = (405-340\gamma),$$

$$\alpha_{16} = (18+71\gamma), \qquad \alpha_{16}' = (-18+71\gamma),$$

$$\alpha_{20} = (6+13\gamma), \qquad \alpha_{20}' = (-6+13\gamma),$$

$$\alpha_{21} = \left(\frac{-3\gamma}{4k_1^2}+\frac{33}{4l_1^2}\right), \qquad \alpha_{21}' = \left(\frac{3\gamma}{4k_1^2}+\frac{33}{4l_1^2}\right),$$

$$\alpha_{22} = \left(\frac{3\gamma}{4k_2^2}+\frac{33}{4l_2^2}\right), \qquad \alpha_{22}' = \left(\frac{-3\gamma}{4k_2^2}+\frac{33}{4l_2^2}\right),$$

$$\alpha_{23} = \left(\frac{3}{4}-\frac{3\gamma}{4k_1^2}-\frac{33}{4l_1^2}\right), \qquad \alpha_{23}' = \left(\frac{3}{4}+\frac{3\gamma}{4k_1^2}-\frac{33}{4l_1^2}\right),$$

$$\alpha_{24} = \left(\frac{3}{4}+\frac{3\gamma}{4k_2^2}-\frac{33}{4l_2^2}\right), \qquad \alpha_{24}' = \left(\frac{3}{4}-\frac{3\gamma}{4k_2^2}-\frac{33}{4l_2^2}\right),$$

$$\alpha_{25} = \left(\frac{6+13\gamma}{6\gamma}-\frac{3\gamma}{4k_1^2}-\frac{33}{4l_1^2}\right), \quad \alpha_{25}' = \left(\frac{-6+13\gamma}{6\gamma}+\frac{3\gamma}{4k_1^2}-\frac{33}{4l_1^2}\right),$$

$$\alpha_{26} = \left(\frac{6+13\gamma}{6\gamma}+\frac{3\gamma}{4k_2^2}-\frac{33}{4l_2^2}\right), \quad \alpha_{26}' = \left(\frac{-6+13\gamma}{6\gamma}-\frac{3\gamma}{4k_2^2}-\frac{33}{4l_2^2}\right),$$

$$\alpha_{27} = \left(\frac{9}{2m_1} - \frac{3\gamma}{4k_1^2} - \frac{33}{4l_1^2}\right), \qquad \alpha'_{27} = \left(\frac{9}{2m_1} + \frac{3\gamma}{4k_1^2} - \frac{33}{4l_1^2}\right),$$

$$\alpha_{28} = \left(\frac{9}{2m_2} + \frac{3\gamma}{4k_2^2} - \frac{33}{4l_2^2}\right), \qquad \alpha'_{28} = \left(\frac{9}{2m_2} - \frac{3\gamma}{4k_2^2} - \frac{33}{4l_2^2}\right),$$

$$\alpha_{29} = \left(\frac{3}{4} + \frac{33}{2n_1} - \frac{3\gamma}{4k_1^2} - \frac{33}{4l_1^2}\right), \quad \alpha'_{29} = \left(\frac{3}{4} + \frac{33}{2n_1} + \frac{3\gamma}{4k_1^2} - \frac{33}{4l_1^2}\right),$$

$$\alpha_{30} = \left(\frac{3}{4} + \frac{33}{2n_2} + \frac{3\gamma}{4k_2^2} - \frac{33}{4l_2^2}\right), \quad \alpha'_{30} = \left(\frac{3}{4} + \frac{33}{2n_2} - \frac{3\gamma}{4k_2^2} - \frac{33}{4l_2^2}\right),$$

$$\alpha_{31} = (-6 + 25\gamma), \qquad \alpha'_{31} = (6 + 25\gamma),$$

$$\alpha_{32} = (43 + 60\gamma), \qquad \alpha'_{32} = (43 - 60\gamma),$$

$$\alpha_{33} = (22 + 65\gamma), \qquad \alpha'_{33} = (-22 + 65\gamma),$$

$$\alpha_{34} = 23, \qquad \alpha'_{34} = 23,$$

$$\alpha_{35} = (285 + 200\gamma), \qquad \alpha'_{35} = (285 - 200\gamma),$$

$$\alpha_{36} = (-54 + 53\gamma), \qquad \alpha'_{36} = (54 + 53\gamma),$$

$$\alpha_{37} = (405 + 340\gamma), \qquad \alpha'_{37} = (405 - 340\gamma),$$

$$\alpha_{38} = (18 + 71\gamma), \qquad \alpha'_{38} = (-18 + 71\gamma),$$

$$\alpha_{39} = 65, \qquad \alpha'_{39} = 65,$$

$$\beta_{21} = -\frac{4\beta_{19}}{l_1^2}, \qquad \beta'_{21} = -\frac{4\beta'_{19}}{l_1^2},$$

$$\beta_{22} = -\frac{4\beta_{19}}{l_2^2}, \qquad \beta'_{22} = -\frac{4\beta'_{19}}{l_2^2},$$

$$\beta_{23} = \frac{4\beta_{19}}{l_1^2}, \qquad \beta'_{23} = \frac{4\beta'_{19}}{l_1^2},$$

$$\beta_{24} = \frac{4\beta_{19}}{l_2^2}, \qquad \beta'_{24} = \frac{4\beta'_{19}}{l_2^2},$$

$$\beta_{25} = \frac{4\beta_{19}}{l_1^2} + \frac{\beta_{20}}{6\gamma}, \qquad \beta'_{25} = \frac{4\beta'_{19}}{l_1^2} - \frac{\beta'_{20}}{6\gamma},$$

$$\beta_{26} = \frac{4\beta_{19}}{l_2^2} + \frac{\beta_{20}}{6\gamma}, \qquad \beta'_{26} = \frac{4\beta'_{19}}{l_2^2} - \frac{\beta'_{20}}{6\gamma},$$

$$\beta_{31} = \frac{4}{3}(4 + \gamma), \qquad \beta'_{31} = \frac{4}{3}(-4 + \gamma),$$

$$\beta_{32} = \frac{-4}{3}(-8+21\gamma), \qquad \beta'_{32} = \frac{-4}{3}(-8-21\gamma),$$

$$\beta_{33} = \frac{4}{3}(2+3\gamma), \qquad \beta'_{33} = \frac{4}{3}(-2+3\gamma),$$

$$\beta_{34} = \frac{-4}{3}(2-9\gamma), \qquad \beta'_{34} = \frac{-4}{3}(2+9\gamma),$$

$$\beta_{35} = \frac{-2}{3}(-87+113\gamma), \qquad \beta'_{35} = \frac{-2}{3}(-87-113\gamma),$$

$$\beta_{36} = \frac{2}{3}(63-\gamma), \qquad \beta'_{36} = \frac{2}{3}(-63-\gamma),$$

$$\beta_{37} = \frac{2}{3}(111-169\gamma), \qquad \beta'_{37} = \frac{2}{3}(111+169\gamma),$$

$$\beta_{38} = \frac{2}{3}(21+5\gamma), \qquad \beta'_{38} = \frac{2}{3}(-21+5\gamma),$$

$$\beta_{39} = \frac{2}{3}(-29+91\gamma), \qquad \beta'_{39} = \frac{2}{3}(-29-91\gamma),$$

$$\gamma'_3 = \frac{9}{16}(1-\gamma), \qquad \gamma'_4 = \frac{7+15\gamma}{1+\gamma},$$

$$\gamma'_5 = -\frac{(15+7\gamma)}{1+\gamma},$$

$$\gamma'_6 = \left\{\frac{15}{4} - \frac{47}{4}\gamma + \frac{1-7\gamma}{2(1-\mu)}\right\},$$

$$\gamma'_7 = -\left\{\frac{87}{4} - \frac{15}{4}\gamma - \frac{11\gamma+3}{2(1-\mu)}\right\},$$

$$\gamma'_8 = \left\{\frac{1}{4} + \frac{15}{4}\gamma + \frac{3+11\gamma}{2(1-\mu)}\right\},$$

$$\gamma'_9 = -7 - \frac{15}{2}\gamma + \frac{1}{2(1-\mu)}(37+25\gamma),$$

$$\gamma'_{10} = 75 - \frac{435}{2}\gamma + \frac{1}{2(1-\mu)}(41-75\gamma),$$

$$\gamma'_{11} = -76 + \frac{55}{2}\gamma + \frac{1}{2(1-\mu)}(41+45\gamma),$$

$$\gamma'_{12} = \frac{75}{2}\gamma + \frac{1}{2(1-\mu)}(1+45\gamma),$$

$$\gamma'_{13} = \frac{1}{4}\left\{1485 - 4025\gamma + \frac{2}{(1-\mu)}(285 - 115\gamma)\right\},$$

$$\gamma'_{14} = -\frac{1}{4}\left\{-567 + 291\gamma + \frac{2}{(1-\mu)}(-171 - 43\gamma)\right\},$$

$$\gamma'_{15} = \frac{1}{4}\left\{2325 - 5665\gamma + \frac{2}{(1-\mu)}(345 - 215\gamma)\right\},$$

$$\gamma'_{16} = \frac{1}{4}\left\{-147 + 111\gamma + \frac{2}{(1-\mu)}(69 + 37\gamma)\right\},$$

$$\gamma'_{17} = -\frac{1}{4}\left\{175 - 1235\gamma + \frac{2}{(1-\mu)}(55 - 185\gamma)\right\},$$

$$\gamma'_{18} = -\frac{3}{64}\left\{15 - 47\gamma + \frac{2}{(1-\mu)}(1 - 7\gamma)\right\},$$

$$\gamma'_{19} = \frac{3}{64}\left\{-1 - 15\gamma - \frac{2}{(1-\mu)}(3 + 11\gamma)\right\},$$

$$\gamma'_{20} = -\frac{1}{4}\left\{87 - 15\gamma - \frac{2}{(1-\mu)}(3 + 11\gamma)\right\},$$

$$\gamma'_{21} = \frac{-3(1-\gamma)}{2k_1^2} - \frac{4\gamma'_{19}}{l_1^2},$$

$$\gamma'_{22} = \frac{3(1-\gamma)}{2k_2^2} - \frac{4\gamma'_{19}}{l_2^2},$$

$$\gamma'_{23} = \frac{-3(1-\gamma)}{2k_1^2} + \frac{4\gamma'_{19}}{l_1^2},$$

$$\gamma'_{24} = \frac{3(1-\gamma)}{2k_2^2} + \frac{4\gamma'_{19}}{l_2^2},$$

$$\gamma'_{25} = \frac{-3(1-\gamma)}{2k_1^2} + \frac{4\gamma'_{19}}{l_1^2} + \frac{\gamma'_{20}}{6\gamma},$$

$$\gamma'_{26} = \frac{3(1-\gamma)}{2k_2^2} + \frac{4\gamma'_{19}}{l_2^2} + \frac{\gamma'_{20}}{6\gamma},$$

$$\gamma'_{31} = -7 - \frac{15}{2}\gamma + \frac{1}{2(1-\mu)}(37 + 25\gamma),$$

$$\gamma'_{32} = 75 - \frac{435}{2}\gamma + \frac{1}{2(1-\mu)}(41 - 75\gamma),$$

$$\gamma'_{33} = -76 + \frac{55}{2}\gamma + \frac{1}{2(1-\mu)}(41 + 45\gamma),$$

$$\gamma'_{34} = \frac{75}{2}\gamma + \frac{1}{2(1-\mu)}(1 + 45\gamma),$$

$$\gamma'_{35} = \frac{1}{4}(1485 - 4025\gamma + \frac{2}{(1-\mu)}(285 - 115\gamma)),$$

$$\gamma'_{36} = \frac{1}{4}(567 - 291\gamma + \frac{2}{(1-\mu)}(171 + 43\gamma)),$$

$$\gamma'_{37} = \frac{1}{4}(2325 - 5665\gamma + \frac{2}{(1-\mu)}(345 - 215\gamma)),$$

$$\gamma'_{38} = -\frac{1}{4}(147 - 111\gamma + \frac{2}{(1-\mu)}(-69 - 37\gamma)),$$

$$\gamma'_{39} = -\frac{1}{4}(175 - 1235\gamma + \frac{2}{(1-\mu)}(55 - 185\gamma)),$$

Appendix B

The coefficients $h_{\alpha_1\alpha_2\beta_1\beta_2}$ (upto order four) are given by

$$h_{3000} = \frac{\sqrt{3}}{32} j_{21}^3 f_{14},$$

$$h_{0300} = \frac{\sqrt{3}}{32} j_{22}^3 f_{14},$$

$$h_{0030} = \frac{\sqrt{3}}{32} j_{23}^3 f_{14} + \frac{1}{32} j_{13}^3 f_{11} + \frac{\sqrt{3}}{32} j_{13}^2 j_{23} f_{11} + \frac{\sqrt{3}}{32} j_{13}^2 j_{23} f_{11} - \frac{3}{32} j_{13} j_{23}^2 f_{13},$$

$$h_{0003} = \frac{\sqrt{3}}{32} j_{24}^3 f_{14} + \frac{1}{32} j_{14}^3 f_{11} + \frac{\sqrt{3}}{32} j_{14}^2 j_{24} f_{11} + \frac{\sqrt{3}}{32} j_{13}^2 j_{24} f_{11} - \frac{3}{32} j_{14} j_{23}^2 f_{14},$$

$$h_{1200} = \frac{3\sqrt{3}}{32} j_{21} j_{22}^2 f_{14},$$

$$h_{1020} = \frac{\sqrt{3}}{32} j_{13}^2 j_{21} f_{12} - \frac{3}{16} j_{13} j_{21} j_{23} f_{13} + \frac{3\sqrt{3}}{32} j_{21} j_{23}^2 f_{14},$$

$$h_{1002} = \frac{\sqrt{3}}{32} j_{14}^2 j_{21} f_{12} - \frac{3}{16} j_{14} j_{21} j_{24} f_{13} + \frac{3\sqrt{3}}{32} j_{21} j_{24}^2 f_{14},$$

$$h_{2100} = \frac{3\sqrt{3}}{32} j_{21}^2 j_{22} f_{14},$$

$$h_{2010} = \frac{-3}{32} j_{13} j_{21}^2 f_{14} + \frac{3\sqrt{3}}{32} j_{21}^2 j_{23} f_{14},$$

$$
\begin{aligned}
h_{2001} &= \frac{-3}{32}j_{14}j_{21}^2f_{14} + \frac{3\sqrt{3}}{32}j_{21}^2j_{24}f_{14},\\
h_{0120} &= \frac{\sqrt{3}}{32}j_{13}^2j_{22}f_{12} - \frac{3}{16}j_{13}j_{22}j_{23}f_{13} + \frac{3\sqrt{3}}{32}j_{22}j_{23}^2f_{14},\\
h_{0102} &= \frac{\sqrt{3}}{32}j_{14}^2j_{22}f_{12} - \frac{3}{16}j_{14}j_{22}j_{23}f_{13} + \frac{3\sqrt{3}}{32}j_{22}j_{24}^2f_{14},\\
h_{0012} &= \frac{3}{32}j_{13}j_{14}^2f_{11} + \left(\frac{\sqrt{3}}{16}j_{13}j_{14}j_{24}f_{13} + \frac{\sqrt{3}}{32}j_{13}j_{24}^2 + \frac{3}{16}j_{14}j_{24}j_{23}\right)f_{13}\\
&\quad + \frac{3\sqrt{3}}{32}j_{23}j_{24}^2f_{14},\\
h_{0021} &= \left(\frac{\sqrt{3}}{16}j_{14}j_{13}j_{23} + \frac{\sqrt{3}}{32}j_{13}^2j_{24}\right)f_{12} + \frac{3}{32}j_{13}^2j_{14}f_{11}\\
&\quad - \left(\frac{3}{16}j_{13}j_{23}j_{24} + \frac{3}{32}j_{14}j_{23}^2\right)f_{13} + \frac{3\sqrt{3}}{32}j_{23}^2j_{24}f_{14},\\
h_{1110} &= -\frac{3}{16}j_{13}j_{21}j_{22}f_{13} + \frac{3\sqrt{3}}{16}j_{21}j_{22}j_{23}f_{14},\\
h_{1101} &= -\frac{3}{16}j_{14}j_{21}j_{22}f_{13} + \frac{3\sqrt{3}}{16}j_{21}j_{22}j_{24}f_{14},\\
h_{1011} &= -\left(\frac{3}{16}j_{14}j_{21}j_{23} + \frac{3}{16}j_{21}j_{13}j_{24}\right)f_{13},\\
&\quad + \frac{\sqrt{3}}{16}j_{13}j_{14}j_{21}f_{12} + \frac{3\sqrt{3}}{16}j_{21}j_{23}j_{24}f_{14}\\
h_{0111} &= \frac{\sqrt{3}}{16}j_{13}j_{14}j_{22}f_{12} - \frac{3}{16}(j_{13}j_{22}j_{24} + j_{14}j_{22}j_{23})f_{13} + \frac{3\sqrt{3}}{16}j_{22}j_{23}j_{24}f_{14},\\
h_{0210} &= \frac{-3}{32}j_{13}j_{22}^2f_{13} + \frac{3\sqrt{3}}{32}j_{22}^2j_{23}f_{14},\\
h_{0201} &= \frac{-3}{32}j_{14}j_{22}^2f_{13} + \frac{3\sqrt{3}}{32}j_{22}^2j_{24}f_{14},\\
h_{0040} &= \frac{1}{256}j_{13}^4(74 + f_{27}) - \frac{5}{192}\sqrt{3}j_{13}^3j_{23}(30\gamma f_{28}) - \frac{3}{128}j_{13}^2j_{23}^2(82 + f_{29})\\
&\quad + \frac{5}{64}\sqrt{3}j_{13}j_{23}^3(18\gamma + f_{30}) - \frac{3}{256}j_{23}^4(2 + f_{31}),\\
h_{4000} &= -\frac{3}{256}(2 + f_{31})j_{21}^4,\\
h_{0022} &= \frac{3}{128}j_{14}^2j_{13}^2(74 + f_{27}) - \frac{5\sqrt{3}}{64}(j_{14}j_{13}^2j_{24} + j_{14}^2j_{13}j_{23})(30\gamma + f_{28})\\
&\quad - \frac{3}{128}(j_{13}^2j_{24}^2 + j_{14}^2j_{23}^2 + 4j_{13}j_{14}j_{23}j_{24})(82 + f_{29})\\
&\quad + \frac{15\sqrt{3}}{64}(j_{14}j_{23}^2j_{24} + j_{24}^2j_{13}j_{23})(18\gamma + f_{30}) - \frac{9}{128}j_{24}^2j_{23}^2(2 + f_{31}),
\end{aligned}
$$

$$
\begin{aligned}
h_{2200} &= -\frac{9}{128}(2+f_{31})j_{22}^2 j_{21}^2,\\
h_{0220} &= -\frac{3}{128}j_{22}^2 j_{13}^2(82+f_{29}) + \frac{15\sqrt{3}}{64}j_{22}^2 j_{13}j_{23}(18\gamma+f_{30}) - \frac{9}{128}j_{22}^2 j_{23}^2(2+f_{31}),\\
h_{2002} &= -\frac{3}{128}j_{21}^2 j_{14}^2(82+f_{29}) + \frac{15\sqrt{3}}{64}j_{21}^2 j_{14}j_{24}(18\gamma+f_{30}) - \frac{9}{128}j_{21}^2 j_{24}^2(2+f_{31}),\\
h_{0004} &= \frac{1}{256}j_{14}^4(74+f_{27}) - \frac{5}{192}\sqrt{3}j_{14}^3 j_{24}(30\gamma+f_{28}) - \frac{3}{128}j_{14}^2 h_{24}^2(82+f_{29})\\
&\quad +\frac{5}{64}\sqrt{3}j_{14}j_{24}^3(18\gamma+f_{30}) - \frac{3}{256}j_{24}^4(2+f_{31}),\\
h_{0400} &= -\frac{3}{256}j_{22}^2(2+f_{31}),\\
h_{0202} &= -\frac{3}{128}j_{22}^2 j_{14}^2(82+f_{29}) + \frac{15\sqrt{3}}{64}j_{22}^2 j_{14}j_{24}(18\gamma+f_{30}) - \frac{9}{128}j_{21}^2 j_{24}^2(2+f_{31}),\\
h_{0013} &= \frac{1}{64}j_{14}^3 j_{13}(74+f_{27}) - \frac{5\sqrt{3}}{192}(j_{14}^3 j_{23} + 3j_{14}^2 j_{13}j_{24})(30\gamma+f_{28})\\
&\quad -\frac{3}{64}(j_{23}j_{24}j_{14}^2 + j_{13}j_{14}j_{24}^2)(82+f_{29}) + \frac{5\sqrt{3}}{64}(3j_{14}j_{24}^2 j_{23} + j_{24}^3 j_{13})(18\gamma+f_{30})\\
&\quad -\frac{3}{64}j_{24}^3 j_{23}(2+f_{31}),\\
h_{1300} &= -\frac{3}{64}j_{22}^3 j_{21}(2+f_{31}),\\
h_{1102} &= -\frac{3}{64}j_{22}j_{21}j_{14}^2(82+f_{29}) + \frac{15\sqrt{3}}{32}j_{22}j_{21}j_{14}j_{24}(18\gamma+f_{30})\\
&\quad -\frac{9}{64}j_{22}j_{21}j_{24}^2(2+f_{31}),\\
h_{0211} &= -\frac{3}{64}j_{14}j_{13}j_{22}^2(82+f_{29}) + \frac{15\sqrt{3}}{32}(j_{14}j_{23} + j_{13}j_{24}j_{22}^2)(18\gamma+f_{30})\\
&\quad -\frac{9}{64}j_{23}j_{24}j_{22}^2(2+f_{31}),\\
h_{0112} &= -\frac{5\sqrt{3}}{192}(j_{14}^3 j_{23} + 3j_{14}^2 j_{13}j_{24})(30\gamma+f_{28}) - \frac{3}{64}(j_{23}j_{24}j_{14}^2 + j_{13}j_{14}j_{24}^2)(82+f_{29})\\
&\quad +\frac{15\sqrt{3}}{64}(2j_{14}j_{22}j_{24}j_{23} + j_{24}^2 j_{22}j_{13})(18\gamma+f_{30}) - \frac{9}{64}j_{24}^2 j_{22}j_{23}(2+f_{31}),\\
h_{1003} &= -\frac{5}{192}\sqrt{3}j_{14}^3 j_{21}(30\gamma+f_{28}) - \frac{3}{64}j_{14}^2 j_{24}j_{21}(82+f_{29})\\
&\quad +\frac{15}{64}\sqrt{3}j_{21}j_{14}j_{24}^2(18\gamma+f_{30}) - \frac{3}{64}j_{21}j_{24}^3(2+f_{31}),\\
h_{1201} &= \frac{15\sqrt{3}}{64}j_{22}^2 j_{21}j_{14}(18\gamma+f_{30}) - \frac{9}{64}j_{22}^2 j_{21}j_{24}(2+f_{31}),\\
h_{0310} &= \frac{5\sqrt{3}}{64}j_{22}^3 j_{13}(18\gamma+f_{30}) - \frac{3}{64}j_{22}^3 j_{23}(2+f_{31})
\end{aligned}
$$

where

$$\begin{aligned}
f_{31} &= (14\gamma + \alpha_{31}A_1 + \alpha'_{31}A'_1 + \gamma'_{31}A'_2 + \beta_{31}P + \beta'_{31}P'),\\
f_{32} &= (6 + \alpha_{32}A_1 + \alpha'_{32}A'_1 + \gamma'_{32}A'_2 + \beta_{32}P + \beta'_{32}P'),\\
f_{33} &= (14\gamma + \alpha_{33}A_1 + \alpha'_{33}A'_1 + \gamma_{33}A_2 + \gamma'_{33}A'_2 + \beta_{33}P + \beta'_{33}P'),\\
f_{34} &= (6 + \alpha_{34}A_1 + \alpha'_{34}A'_1 + \gamma'_{34}A'_2 + \beta_{34}P + \beta'_{34}P'),\\
f_{35} &= (\alpha_{35}A_1 + \alpha'_{35}A'_1 + \gamma'_{35}A'_2 + \beta_{35}P + \beta'_{35}P'),\\
f_{36} &= (\alpha_{36}A_1 + \alpha'_{36}A'_1 + \gamma'_{36}A'_2 + \beta_{36}P + \beta'_{36}P'),\\
f_{37} &= (\alpha_{37}A_1 + \alpha'_{37}A'_1 + \gamma'_{37}A'_2 + \beta_{37}P + \beta'_{37}P'),\\
f_{38} &= (\alpha_{38}A_1 + \alpha'_{38}A'_1 + \gamma'_{38}A'_2 + \beta_{38}P + \beta'_{38}P'),\\
f_{39} &= (\alpha_{39}A_1 + \alpha'_{39}A'_1 + \gamma'_{39}A'_2 + \beta_{39}P + \beta'_{39}P'),
\end{aligned}$$

and α_i's and β_i's are given in Appendix A.

Appendix C

The coefficients $x_{\alpha_1\alpha_2\beta_1\beta_2}$ and $y_{\alpha_1\alpha_2\beta_1\beta_2}$ (upto order three) are given by

$$\begin{aligned}
x_{0120} &= \frac{-1}{2}h_{0021}\omega_2 + \frac{1}{2}\frac{h_{2001}}{\omega_2^2}\omega_2 + \frac{1}{2}\frac{h_{1110}}{\omega_1},\\
h_{0120} &= \frac{-1}{2}\frac{h_{1011}}{\omega_1}\omega_2 + \frac{1}{2}\frac{h_{2100}}{\omega_1^2} - \frac{1}{2}h_{0120},\\
y_{0012} &= \frac{h_{0111}}{\omega_2} - \frac{h_{1200}}{\omega_1} + \frac{h_{1200}}{\omega_1}\omega_2^2,\\
x_{0012} &= -h_{0012} - \frac{h_{1101}}{\omega_1\omega_2} + \frac{h_{0210}}{\omega_2^2},\\
y_{1002} &= \frac{1}{2}\frac{h_{1101}}{\omega_2} + \frac{1}{2}\frac{h_{0210}}{\omega_2^2}\omega_1 - \frac{1}{2}h_{0012}\omega_1,\\
x_{1002} &= -\frac{1}{2}\frac{h_{0111}}{\omega_2}\omega_1 - \frac{1}{2}h_{1002} + \frac{1}{2}\frac{h_{1200}}{\omega_2^2},\\
x_{1011} &= \frac{-h_{2001}}{\omega_1} - h_{0021}\omega_1,\\
y_{1011} &= \frac{h_{0120}}{\omega_2} + \frac{h_{2100}}{\omega_1}\omega_2,\\
x_{0201} &= -\frac{3}{4}\frac{h_{0300}}{\omega_2} - \frac{1}{4}h_{0102}\omega_2,
\end{aligned}$$

$$
\begin{aligned}
y_{0201} &= -\frac{1}{4}h_{0201} + \frac{3}{4}h_{0003}\omega_2^2\,, \\
x_{0003} &= \frac{h_{0300}}{\omega_2^3} - \frac{h_{0102}}{\omega_2}\,, \\
y_{0003} &= \frac{h_{0201}}{\omega_2^2} - h_{0003}\,, \\
x_{0111} &= \frac{h_{1200}}{\omega_1\omega_2} + \frac{h_{1002}}{\omega_1}\omega_2\,, \\
y_{0111} &= -h_{0012}\omega_2 - \frac{h_{0210}}{\omega_2}\,, \\
x_{0030} &= -\frac{h_{2010}}{\omega_1^2} + h_{0030}\,, \\
y_{0030} &= -\frac{h_{3000}}{\omega_1^3} + \frac{h_{1020}}{\omega_1}\,, \\
x_{1020} &= -\frac{1}{2}h_{1020} - \frac{3}{2}\frac{h_{3000}}{\omega_1^2}\,, \\
y_{1020} &= \frac{3}{2}h_{0030}\omega_1 + \frac{1}{2}\frac{h_{2010}}{\omega_1}\,, \\
x_{0021} &= \frac{h_{0120}}{\omega_2} - \frac{h_{2100}}{(\omega_1^2\omega_2)} - \frac{h_{1011}}{\omega_1}\,, \\
y_{0021} &= h_{0021} - \frac{h_{2001}}{\omega_1^2} + \frac{h_{1110}}{\omega_1}\omega_2\,,
\end{aligned}
$$

the remaining ten coefficient are given by the formula

$$
h'_{\alpha_1\alpha_2\beta_1\beta_2} = (x_{\alpha_1\alpha_2\beta_1\beta_2} + y_{\alpha_1\alpha_2\beta_1\beta_2})\left(-\frac{\omega_1}{2}\right)^{\alpha_1-\beta_1}\left(\frac{\omega_2}{2}\right)^{\alpha_2-\beta_2}.
$$

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