

The Radio Sky: Problem Sheet 3

IUCAA-NCRA Graduate School

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- These problems are for your own practice and will not be graded. They are designed to help you prepare for the mid-term and final examinations. However, I strongly encourage you to ask questions and discuss the solutions.
 - If you spot any potential errors or find a question unclear, please do not hesitate to let me know.
 - You are welcome to consult books, online resources, and discuss the problems with your peers. The key, however, is to ensure you personally understand the solutions, as this will be vital for your performance in the examinations.
 - If you choose to use notation or conventions that differ from those presented in lectures, please define them clearly at the start and apply them consistently.
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1. Consider a two-level system with lower level a and upper level b (energy levels $E_b > E_a$, transition frequency ν_{ba}), with degeneracies g_a and g_b , respectively.

(a) The critical density (n_{crit}) is the collider density n_{coll} at which the rate of collisional de-excitation equals the rate of spontaneous radiative de-excitation. Find an expression for n_{crit} in terms of the spontaneous radiative de-excitation coefficient A_{ba} and the collisional de-excitation coefficient $C_{ba} \equiv n_{\text{coll}} \langle \sigma_{ba} v \rangle$, where $\langle \sigma_{ba} v \rangle$ is the velocity-averaged collisional cross-section.

(b) Starting with the steady-state equation for the level populations

$$\frac{n_a g_b}{n_b g_a} = \frac{C_{ba} + \frac{c^2}{2h\nu_{ba}^3} A_{ba} I_{\nu_{ba}} + A_{ba}}{C_{ba} e^{-h\nu_{ba}/k_B T_K} + \frac{c^2}{2h\nu_{ba}^3} A_{ba} I_{\nu_{ba}}},$$

rewrite this equation in the Rayleigh-Jeans limit. Assume the radiation field is the CMB, so $I_{\nu_{ba}} \approx B_\nu(T_{\text{CMB}})$.

(c) Show that the general expression for T_{ex} can be expressed in terms of the kinetic temperature T_K , the radiation temperature T_{CMB} , and the ratio of the collider density to the critical density, $n_{\text{coll}}/n_{\text{crit}}$.

(d) What does this new form tell you about the T_{ex} in the very low density ($n_{\text{coll}} \ll n_{\text{crit}}$) and very high density ($n_{\text{coll}} \gg n_{\text{crit}}$) limits?

2. The collisional rate coefficient is defined as

$$q_{ab} = \langle \sigma_{ab} v \rangle = \int_0^\infty dv \sigma_{ab}(v) v f(v),$$

where $f(v)$ is the 3D speed distribution.

Assume a gas of particles (mass m) at a kinetic temperature T_K . The cross-section for a collisional excitation $a \rightarrow b$ (which requires energy $\Delta E = h\nu_{ba}$) is given by a simple “step” model

$$\sigma_{ab}(E) = \begin{cases} 0 & \text{if } E < \Delta E \\ \sigma_0 & \text{if } E \geq \Delta E \end{cases}$$

where $E = mv^2/2$ is the kinetic energy of the particle. Write down and evaluate the integral for $q_{ab} = \langle \sigma_{ab} v \rangle$ using the 3D Maxwell-Boltzmann speed distribution

$$f(v) dv = 4\pi \left(\frac{m}{2\pi k_B T_K} \right)^{3/2} v^2 \exp \left(-\frac{mv^2}{2k_B T_K} \right) dv.$$

Show that the final rate coefficient has the form

$$q_{ab}(T_K) \propto T_K^{1/2} \left(1 + \frac{\Delta E}{k_B T_K} \right) e^{-\Delta E/k_B T_K}.$$

What does the $e^{-\Delta E/k_B T_K}$ term represent physically?

3. Let a hydrogen atom be placed in a uniform magnetic field $\vec{B}$.

(a) Show that the vector potential

$$\vec{A} = \frac{1}{2} (\vec{B} \times \vec{x})$$

produces the correct uniform magnetic field $\vec{B}$. Does this potential satisfy the Coulomb gauge?

(b) Assuming the charge of the electron to be $-e$, show that the non-relativistic Schrödinger equation for the system (assuming Coulomb gauge) is

$$\left[-\frac{\hbar^2}{2m_e} \vec{\nabla}^2 - \frac{e^2}{r} - \frac{i\hbar e}{m_e c} \vec{A} \cdot \vec{\nabla} + \frac{e^2}{2m_e c^2} \vec{A}^2 - \vec{\mu} \cdot \vec{B} \right] \psi(\vec{x}) = E \psi(\vec{x}),$$

where $\vec{\mu}$ is the electron spin angular momentum.

(c) Show that the linear term in $\vec{A}$ can be written as

$$\frac{e}{2m_e c} \vec{B} \cdot \vec{L}.$$

(d) Estimate the order of magnitude of the magnetic field (in Gauss) that is required to make the quadratic term in $\vec{A}$ comparable to the linear term. How does this compare with the typical magnetic fields observed in the Milky Way?

(e) Show that, ignoring the quadratic term, the Schrödinger equation reduces to

$$\left[-\frac{\hbar^2}{2m_e} \vec{\nabla}^2 - \frac{e^2}{r} + \frac{e}{2m_e c} \vec{B} \cdot (\vec{L} + 2\vec{S}) \right] \psi(\vec{x}) = E \psi(\vec{x}).$$

(f) Suppose we treat the magnetic field term as perturbation over the unperturbed Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m_e} \vec{\nabla}^2 - \frac{e^2}{r}.$$

Assuming the magnetic field to be along the z -direction, show that the energy corrections are given by

$$\Delta E_{l,m_l,m_s} = \hbar \omega_L (m_l + 2m_s),$$

where

$$\omega_L = \frac{eB}{2m_e c}$$

is the Larmor angular frequency. What is the energy split for the hydrogen ground state?

4. Let us assume that the relativistic motion of an electron in a hydrogen atom can be described by the Klein-Gordon equation.

- (a) Find the Klein-Gordon equation for an electron moving in the electrostatic potential of an infinitely heavy nucleus (just a proton in the case of hydrogen).
- (b) Assuming we are looking for the stationary state solutions, let us decompose the electron wave function as $\psi(\vec{x}, t) = R(r) Y_{lm}(\theta, \phi) e^{-iEt/\hbar}$. Find the radial equation satisfied by $R(r)$ in a form that resembles the corresponding equation obtained from the non-relativistic Schrödinger equation.
- (c) Compare the radial differential equation with its non-relativistic counterpart and obtain the energy eigenvalues E_{nl} . You do not need to derive the solution, try to obtain them from the knowledge of the energy eigenvalues in the non-relativistic case. Also ensure that the definition of the principal quantum number n is identical to that used in the non-relativistic equation, i.e., n can take values $l + 1, l + 2, \dots$ for a given l .
- (d) Expand the energy eigenvalues obtained above in powers of α^2 (α being the fine structure constant) and interpret the results for the terms corresponding to α^0 , α^2 and α^4 . How do these solutions compare with those obtained from the Dirac equation (as derived in the class)?

5. The Dirac Hamiltonian for a free particle is

$$H = c \vec{\alpha} \cdot \vec{p} + \beta mc^2.$$

- (a) Show that the orbital angular momentum $\vec{L}$ of the particle is not conserved, i.e.,

$$[H, \vec{L}] \neq 0.$$

- (b) Consider the 4×4 spin operator:

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

where $\vec{\sigma}$ are the 2×2 Pauli matrices. Prove that the spin operator $\vec{\Sigma}$ is *also* not conserved, by calculating $[H, \vec{\Sigma}]$.

- (c) Finally, show that the *total* angular momentum

$$\vec{J} = \vec{L} + \frac{\hbar}{2} \vec{\Sigma}$$

is conserved, by proving $[H, \vec{J}] = 0$.

6. The different energy levels of the hydrogen atom are split because of the relativistic corrections. For example, the Lyman series lines are split into a doublet

$$1S_{1/2} \longleftrightarrow nP_{1/2}, \quad 1S_{1/2} \longleftrightarrow nP_{3/2}.$$

Consider the Balmer series lines which connect the $n = 2$ state with higher states $n > 2$.

- (a) Write down all the possible *allowed* transitions (accounting for the fine-structure) between different levels of $n = 2$ and any $n > 2$ states. Please give short justifications (based on selection rules) for each of these transitions.
- (b) What is the number of *distinct* components of the Balmer line for a $2 \longleftrightarrow n$ transition ($n > 2$)?

7. An astronomer is studying a cold atomic hydrogen (HI) cloud. To determine its properties, they conduct a classic “on-off” 21 cm line observation using a single-dish radio telescope.

- “On-Source”: They point the telescope directly at a bright, background continuum source (a quasar) that is behind the cloud. The brightness temperature of the background quasar (measured at a frequency just off the 21 cm line) is $T_{\text{bg}} = 50.0$ K. The brightness temperature at the line-centre, *including the background source*, is $T_{\text{on,peak}} = 40.0$ K. Note that $T_{\text{on,peak}}$ is lower than the off-line continuum level because the cold cloud absorbs some of the quasar’s continuum radiation at the line frequency.
- “Off-Source”: They point the telescope to an adjacent part of the cloud, away from the background quasar. The peak brightness temperature of the emission line is measured to be $T_{\text{off,peak}} = 5.0$ K.

Assume the cloud is uniform over the small angular distance between the two pointings. Both the emission and absorption profiles are well-fit by a Gaussian with the same Full-Width at Half-Maximum (FWHM) of $\Delta v = 10.0$ km/s. The cloud is at a distance $D = 500$ pc and has an angular diameter of $\theta = 1.0^\circ$.

- (a) Calculate the spin temperature (T_s) and peak optical depth (τ_{peak}) using the “on” and “off” measurements.
- (b) Show that the column density of neutral hydrogen N_{HI} is given by the expression

$$N_{\text{HI}} = \left(\frac{32\pi k_B T_s}{3A_{10}ch} \right) \int dv \tau_v$$

where $dv/c = d\nu/\nu_{10}$ is the velocity, ν_{10} is the frequency of the 21 cm line and all other symbols have their usual meanings.

- (c) Using the observed measurements, find the column density of the cloud under consideration.
- (d) Calculate the total HI mass (M_{HI}) of the cloud.

8. The Schrödinger equation for a diatomic molecule, under Born-Oppenheimer approximation, is given by

$$-\frac{\hbar^2}{2\mu} \left[\chi''_{e,v,J}(R) - \frac{J(J+1)}{R^2} \chi_{e,v,J}(R) \right] + E_e(R) \chi_{e,v,J}(R) = E_{e,v,J} \chi_{e,v,J}(R),$$

where $\chi_{e,v,J}(R)$ is the nuclear radial wave function, μ is the reduced mass of the nuclei, R is the distance between the two nuclei, $E_e(R)$ is the eigenvalue of the electronic eigenstate and $E_{e,v,J}$ is the total energy. The characteristic quantum numbers are v, e, J .

- (a) Assuming the rigid rotor model (i.e., $R = R_0$ in the rotational energy term) and approximating the electronic potential as a harmonic oscillator around the equilibrium radius R_0 , show that the total energy of the molecule can be written as

$$E_{e,v,J} = E_e(R_0) + E_J + E_v, \quad E_J = \frac{\hbar^2 J(J+1)}{2\mu R_0^2}, \quad E_v = \hbar\omega_0 \left(v + \frac{1}{2} \right),$$

where $\omega_0 = \sqrt{E''_e(R_0)/\mu}$ is the vibrational frequency.

- (b) Going beyond the rigid rotor model, account for the centrifugal distortion effects by considering the effective potential

$$V_{\text{eff}}(R) = E_e(R) + \frac{\hbar^2 J(J+1)}{2\mu R^2},$$

and expanding it about the new equilibrium position R_J defined by

$$V'_{\text{eff}}(R_J) = 0.$$

Assuming R_J is close to R_0 , show that the rotational energy levels can be approximated as

$$E_J \approx hBJ(J+1) - hDJ^2(J+1)^2$$

where $B = h/(8\pi^2 I_0)$ (with $I_0 = \mu R_0^2$) is the standard rotational constant and D is the centrifugal distortion constant. Find an expression for D in terms of B and ω_0 .

- (c) How does centrifugal distortion affect the observed “rotational ladder”, i.e., the frequencies $\nu_{J+1 \rightarrow J} = (E_{J+1} - E_J)/h$, of a molecule like CO?

9. A radio telescope is pointed at the center of a dense molecular cloud, measuring the peak line brightness *above the background* of the $J = 1 \rightarrow 0$ transition for two isotopologues, ^{12}CO and ^{13}CO . The observed values and assumptions are as follows:

- Peak line brightness (above the background) $T_{\text{line}}(^{12}\text{CO}) = 15.27$ K

- Peak line brightness (above the background) $T_{\text{line}}(^{13}\text{CO}) = 3.60 \text{ K}$
- Background temperature $T_{\text{bg}} = 2.73 \text{ K}$ (CMB)

The intrinsic abundance ratio of the molecules is $[^{12}\text{CO}]/[^{13}\text{CO}] = 70$. The optical depth is proportional to the column density (and thus the abundance). You can assume that the gas density is high enough that both lines are thermalized, so their excitation temperature is equal to the gas kinetic temperature.

- Calculate the kinetic temperature based on the observations and assumptions, and also the optical depth of both lines.
- Why is the observed brightness ratio (4.24) so different from the intrinsic abundance ratio (70)?