The Radio Sky: Problem Sheet 2
IUCAA-NCRA Graduate School

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- These problems are for your own practice and will not be graded. They are designed to help you prepare for the mid-term and final examinations. However, I strongly encourage you to ask questions and discuss the solutions.
- If you spot any potential errors or find a question unclear, please do not hesitate to let me know.
- You are welcome to consult books, online resources, and discuss the problems with your peers. The key, however, is to ensure you personally understand the solutions, as this will be vital for your performance in the examinations.
- If you choose to use notation or conventions that differ from those presented in lectures, please define them clearly at the start and apply them consistently.

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1. Recombination lines occur when an electron recombines with the proton to form the neutral atom. The electron can be captured at a higher state and then it cascades downwards giving rise to different lines.

   (a) Consider the recombination line arising from a transition $n + \Delta n \longrightarrow n$, with $n \gg 1$ and $\Delta n \sim 1$, Show that the frequency of this line scales as $\Delta n / n^3$.

   (b) Determine the frequency of the H 90α line (i.e., the $n = 91 \longrightarrow 90$ transition of hydrogen).

   (c) Radio recombination lines can be used to determine the ratio of helium to hydrogen densities in the ionized regions of the interstellar medium. Let us assume that we can measure, say, the 90α line arising from recombination of ionized hydrogen and also from the recombination of singly-ionized helium. Determine the frequency difference between these two lines and find the equivalent Doppler velocity shift corresponding to this frequency difference.

2. The vector potential for an electromagnetic wave in free space can be expanded as

$$\vec{A}(\vec{x}, t) = \sum_{\vec{k}} \sum_{\alpha} \left[a_\alpha(\vec{k})\, \hat{e}_\alpha(\hat{k})\, \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{x} - \omega t)} + a^*_\alpha(\vec{k})\, \hat{e}^*_\alpha(\hat{k})\, \mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{x} - \omega t)} \right],$$

   where $\hat{e}_\alpha(\hat{k})$ are the polarization vectors (with $\alpha = 1, 2$) orthogonal to the wave vector $\vec{k}$. The sum over $\vec{k}$ is over all wave vectors allowed in a cubic box of volume $V = L^3$ with boundary conditions $k_x, k_y, k_z = 2\pi n_x / L, 2\pi n_y / L, 2\pi n_z / L$ with n_x, n_y, n_z being integers. The angular frequency is given by $\omega = c|\vec{k}|$.

   Derive the expression for Hamiltonian

$$H_{\mathrm{rad}} = \frac{1}{8\pi} \int \mathrm{d}^3x \left(\vec{E}^2 + \vec{B}^2 \right),$$

   and the momentum

$$\vec{P}_{\mathrm{rad}} = \frac{1}{4\pi c} \int \mathrm{d}^3x \left(\vec{E} \times \vec{B} \right),$$

   in terms of the coefficients $a_\alpha(\vec{k})$ and $a^*_\alpha(\vec{k})$.

3. (a) Show that the Lagrangian of a (classical) charged particle of mass m and charge q in presence of an external electromagnetic field is

$$L = \frac{1}{2}m\dot{\vec{x}}^2 - q\phi(\vec{x},t) + q\frac{\dot{\vec{x}}}{c}\cdot\vec{A}(\vec{x},t),$$

where ϕ is the electric scalar potential and $\vec{A}$ is the magnetic vector potential. It is sufficient to show that the above Lagrangian gives the correct equation of motion.

(b) Hence show that the Hamiltonian is

$$H = \frac{1}{2m}\left[\vec{p} - \frac{q}{c}\vec{A}(\vec{x},t)\right]^2 + q\phi(\vec{x},t).$$

(Thus one can incorporate the effects of the electromagnetic field into the Hamiltonian by replacing $\vec{p} \to \vec{p} - (q/c)\vec{A}$ and $H \to H - q\phi$.)

(c) The state of a spin-$1/2$ particle can be written as

$$\langle\vec{x}|\psi\rangle = \begin{pmatrix} \psi_+(\vec{x}) \\ \psi_-(\vec{x}) \end{pmatrix},$$

where the wave functions $\psi_\pm(\vec{x})$ satisfy the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\langle\vec{x}|\psi\rangle}{\partial t} = \langle\vec{x}|H|\psi\rangle\,.$$

In the absence of any external field or potential, this becomes

$$\mathrm{i}\hbar\frac{\partial\langle\vec{x}|\psi\rangle}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\langle\vec{x}|\psi\rangle\,.$$

Show that the above equation is equivalent to

$$\mathrm{i}\hbar\frac{\partial\langle\vec{x}|\psi\rangle}{\partial t} = -\frac{\hbar^2}{2m}\left(\vec{\sigma}\cdot\vec{\nabla}\right)^2\langle\vec{x}|\psi\rangle\,,$$

where the components of $\vec{\sigma}$ are the 2×2 Pauli matrices. This is a simple way of introducing spin (by hand) into the non-relativistic Schrödinger equation.

(d) Argue that in the presence of electromagnetic fields, the above equation modifies to

$$\mathrm{i}\hbar\frac{\partial\langle\vec{x}|\psi\rangle}{\partial t} = \frac{1}{2m}\left[\vec{\sigma}\cdot\left(-\mathrm{i}\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)\right]^2\langle\vec{x}|\psi\rangle + q\phi\,\langle\vec{x}|\psi\rangle\,,$$

where $\vec{A}$ and ϕ should be treated as quantum mechanical operators (in the coordinate basis).

(e) Show that the above equation can be written as

$$\mathrm{i}\hbar\frac{\partial\langle\vec{x}|\psi\rangle}{\partial t} = \frac{1}{2m}\left(-\mathrm{i}\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2\langle\vec{x}|\psi\rangle - \frac{q\hbar}{2mc}\vec{\sigma}\cdot\vec{B}\,\langle\vec{x}|\psi\rangle + q\phi\,\langle\vec{x}|\psi\rangle\,,$$

where $\vec{B} = \vec{\nabla}\times\vec{A}$ is the magnetic field. Can you interpret the significance of this equation?

4. Instead of the treating the electromagnetic field quantum mechanically, let us work out the transition rates when the field is *classical*.

(a) In the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, the vector potential $\vec{A}$ satisfies the wave equation, and hence can be expanded as

$$\vec{A}(\vec{x},t) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[a(\vec{k})\,\hat{\epsilon}(\vec{k})\,\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)} + \text{ c.c.}\right],\quad \omega = k\,c,$$

where $\hat{\epsilon}(\vec{k})$ represents the direction of the polarization. Calculate the energy density

$$u = \frac{1}{8\pi V}\int\mathrm{d}^3x\left[\vec{E}^2(\vec{x},t) + \vec{B}^2(\vec{x},t)\right]$$

in the radiation field and show that the specific intensity is given by

$$I_\omega(\hat{n}) = \frac{1}{V}\left(\frac{\omega}{2\pi c}\right)^4|a(\vec{k})|^2.$$

(b) The interaction between an atom and the radiation field can be described by the Hamiltonian

$$H_1 = \frac{e}{m_e c}\vec{A}\cdot\vec{P} = -\frac{\mathrm{i}e\hbar}{m_e c}\vec{A}\cdot\vec{\nabla}.$$

Show that H_1 can be written as

$$H_1 = V\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[H^{\mathrm{abs}}(\vec{k})\,\mathrm{e}^{-\mathrm{i}\omega t} + H^{\mathrm{emi}}(\vec{k})\,\mathrm{e}^{\mathrm{i}\omega t}\right].$$

Write down the expressions for $H^{\mathrm{abs}}(\vec{k})$ and $H^{\mathrm{emi}}(\vec{k})$.

(c) Using the expressions for Fermi's golden rule

$$\begin{aligned}
R^{\mathrm{abs}}_{a\to b} &= \frac{2\pi}{\hbar^2}\left|\langle E_b|H^{\mathrm{abs}}|E_a\rangle\right|^2\delta_D(\omega-\omega_{ba}),\\
R^{\mathrm{emi}}_{b\to a} &= \frac{2\pi}{\hbar^2}\left|\langle E_a|H^{\mathrm{emi}}|E_b\rangle\right|^2\delta_D(\omega-\omega_{ba}),
\end{aligned}$$

show that the absorption rate for the radiation-matter interaction is

$$R^{\mathrm{abs}}_{a\to b} = \frac{4\pi^2 e^2}{m_e^2 c\,\omega_{ba}^2}\int \mathrm{d}\Omega\; I_\omega(\hat{n})\,|\mathcal{M}_{ba}(\vec{k})|^2,$$

where

$$\mathcal{M}_{ba}(\vec{k}) = \int \mathrm{d}^3x\;\psi_b^*(\vec{x})\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\,\hat{\epsilon}(\vec{k})\cdot\vec{\nabla}\psi_a(\vec{x}).$$

(d) Find out the corresponding emission rate and show that $R^{\mathrm{emi}}_{b\to a} = R^{\mathrm{abs}}_{a\to b}$

(e) What is main difference you see between the classical and quantum treatments of the radiation field?

5. Consider a free electron (ignoring spin) of charge $-e$ moving in a uniform time-independent magnetic field $\vec{B}$.

(a) Show that the vector potential

$$\vec{A} = \frac{1}{2}\left(\vec{B}\times\vec{x}\right)$$

produces the correct uniform magnetic field $\vec{B}$.

(b) Now orient the axes such that the magnetic field is in the z-direction, i.e., $\vec{B} = B\hat{z}$. Calculate the vector potential for this case.

(c) Use the gauge variance of $\vec{B}$ under the transformation $\vec{A}\to\vec{A}+\vec{\nabla}\varphi$ to show that the quantity

$$\vec{A} = -B\,y\,\hat{x}$$

too is an appropriate vector potential. What φ did you choose to obtain the new vector potential from the old one (give your answer up to an additive constant)?

(d) Show that the Hamiltonian for this system can be written as

$$H = \frac{1}{2m_e}\left(P_x - \frac{e}{c}By\right)^2 + \frac{1}{2m_e}(P_y^2 + P_z^2).$$

(e) Show that $[H, P_x] = [H, P_z] = 0$.

(f) Given the above commutation relations, we can choose the wave function $\psi(\vec{x})$ to be a simultaneous eigenfunction of H, P_x, P_z. Let $\psi_{p_x}(x)$ be the eigenfunction of P_x with eigenvalue p_x. Obtain the explicit form of $\psi_{p_x}(x)$. Similarly, write down the explicit form of the eigenfunction $\psi_{p_z}(z)$ of P_z. What are the allowed ranges of the eigenvalues p_x and p_z?

(g) Try a solution of the form

$$\psi(\vec{x}) = \psi_{p_x}(x)\,\psi_{p_z}(z)\,\xi(y),$$

and show that the Schrödinger equation $H\psi = E\psi$ reduces to an equation which resembles a simple harmonic oscillator system in the y-direction. Hence show that the energy eigenvalues are given by

$$E_n = (2n+1)\hbar\omega_L + \frac{p_z^2}{2m_e},\ n = 0, 1, 2, \ldots,$$

where

$$\omega_L = \frac{eB}{2m_e c}$$

is the Larmor angular frequency.

6. In the time-dependent perturbation theory, we start with a Hamiltonian of the form

$$H = H_0 + \epsilon H_1(t),$$

where ϵ is assumed to be a small number. We then assume solutions of the form

$$|\psi(t)\rangle = \sum_i c_i(t)\,|E_i\rangle\ \mathrm{e}^{-\mathrm{i}E_i t/\hbar},$$

expand the coefficients as

$$c_i(t) = c_i^{(0)}(t) + \epsilon c_i^{(1)}(t) + \epsilon^2 c_i^{(2)}(t) + \ldots$$

put it back into the Schrödinger equation and obtain the perturbation solutions by equating terms with equal powers of ϵ. For definiteness, we assume that the perturbation turns on at $t = 0$, hence $c_i(0) = c_i^{(0)}$ (the unperturbed solution). We also assume that the system initially is on one of the stationary eigenstates, say, the ath one

$$c_i(0) = c_i^{(0)} = \delta_{ia}.$$

All other coefficients $c_i^{(0)},\ i \neq a$ vanish.

(a) Find the coefficients $c_i^{(2)}(t)$ in the second order of perturbation. Interpret the solution physically. Write down the solution $c_i(t)$ correct to the second order.

(b) Let the perturbed potential be such that $\langle E_a|H_1(t)|E_a\rangle = 0$. Find the solution for the state $i = a$ (i.e., the state in which the system was initially in) correct up to the second order. Interpret the result.

(c) Consider the scattering process

$$a + \gamma \longrightarrow b + \gamma',$$

where an incident photon γ interacts with an atom in an initial state a resulting in the atom in state b and a scattered photon γ'. The interaction, as we have seen in the lectures, is described by an Hamiltonian

$$H = H_0 + H_1 + H_2,$$

where H_0 is the unperturbed Hamiltonian in the absence of any interaction, H_1 corresponds to the matter-radiation interaction and H_2 is the two-photon interaction.

Argue that the scattering amplitude vanishes in the first order perturbation when only H_1 is considered.

If we consider the second order perturbation to find the lowest non-vanishing amplitude for H_1, can we ignore the H_2 term? Explain your answer.

7. Consider a transition between two states $|a\rangle$ and $|b\rangle$, where the transition dipole moment $\vec{X}_{ba}$ is aligned with the z-axis (i.e., $\vec{X}_{ba} = X_{ba}\hat{z}$). An incoming plane wave of radiation propagates along the x-axis ($\vec{k} = k\hat{x}$).

(a) Calculate the relative absorption rate for the case where the radiation is linearly polarized along the z-axis ($\hat{e} = \hat{z}$).

(b) Calculate the relative absorption rate for the case where the radiation is linearly polarized along the y-axis ($\hat{e} = \hat{y}$).

(c) Now consider right-circularly polarized (RCP) light. The polarization vector can be written as $\hat{e}_{\mathrm{RCP}} = (\hat{y} + \mathrm{i}\hat{z})/\sqrt{2}$. Calculate the relative absorption rate for this case.

(d) Based on your results, explain how observing the absorption of polarized background radiation can be used to probe the orientation of atoms (or molecules) in space. This is a principle behind studying interstellar magnetic fields, where magnetic fields can align atoms/molecules.