

The Radio Sky: Problem Sheet 1

IUCAA-NCRA Graduate School

October – December 2025

16 October 2025

-
- These problems are for your own practice and will not be graded. They are designed to help you prepare for the mid-term and final examinations. However, I strongly encourage you to ask questions and discuss the solutions.
 - If you spot any potential errors or find a question unclear, please do not hesitate to let me know.
 - You are welcome to consult books, online resources, and discuss the problems with your peers. The key, however, is to ensure you personally understand the solutions, as this will be vital for your performance in the examinations.
 - If you choose to use notation or conventions that differ from those presented in lectures, please define them clearly at the start and apply them consistently.
-

1. Starting from the Maxwell's equations

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, & \vec{\nabla} \cdot \vec{B} &= 0, \\ \vec{\nabla} \cdot \vec{E} &= 4\pi\rho_e, & \vec{\nabla} \times \vec{B} &= \frac{4\pi}{c} \vec{j}_e + \frac{1}{c} \frac{\partial \vec{E}}{\partial t},\end{aligned}$$

show that the potentials ϕ and $\vec{A}$, defined as,

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}.$$

satisfy the equations

$$\begin{aligned}\vec{\nabla}^2 \phi &= -4\pi\rho_e - \frac{1}{c} \frac{\partial (\vec{\nabla} \cdot \vec{A})}{\partial t}, \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\frac{4\pi}{c} \vec{j}_e + \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right).\end{aligned}$$

These equations can be simplified by choosing an appropriate gauge. In the Coulomb gauge one chooses $\vec{\nabla} \cdot \vec{A} = 0$. Write down the equations satisfied by the potentials under this condition.

Hence show that they are related to the fields by

$$\begin{aligned}\phi(\vec{x}, t) &= \int d^3x' \frac{\rho_e(\vec{x}', t)}{|\vec{x} - \vec{x}'|}, \\ \vec{A}(\vec{x}, t) &= \frac{1}{4\pi} \vec{\nabla} \times \int d^3x' \frac{\vec{B}(\vec{x}', t)}{|\vec{x} - \vec{x}'|}.\end{aligned}$$

2. Show that the radiative transfer equation

$$\frac{dI_\nu}{ds} = j_\nu - \kappa_\nu I_\nu,$$

admits a general solution of the form

$$I_\nu(s) = I_\nu(0) e^{-\tau_\nu(s)} + \int_0^s ds' j_\nu(s') e^{-[\tau_\nu(s) - \tau_\nu(s')]},$$

where $\tau_\nu(s) = \int_0^s ds' \kappa_\nu(s')$ is the optical depth at frequency ν along the path s and $I_\nu(0)$ is the specific intensity at $s = 0$.

3. A compact HII region (a cloud of ionized hydrogen) has a constant thermodynamic temperature of $T = 8000$ K. An observer measures its specific intensity at two different frequencies:

- A radio observation at frequency 1.4 GHz.
- A mid-infrared observation at wavelength 1.5 microns.

For both frequencies, calculate the expected specific intensity I_ν assuming the cloud is a perfect black body. Then, calculate the brightness temperature T_b an observer would deduce at each frequency using the Rayleigh-Jeans formula. Compare T_b with the true temperature T in both cases and explain any differences.

4. Consider a spherical cloud of gas of radius R , located at a distance d from an observer. The cloud is composed of a uniform medium with a constant emission coefficient j_ν , absorption coefficient κ_ν , and source function $S_\nu = j_\nu / \kappa_\nu$. An observer measures the flux density from this cloud.

- (a) By solving the radiative transfer equation along different lines of sight and integrating over the source's solid angle, show that the flux density received by the observer is given by the general expression:

$$F_\nu = \pi S_\nu \left(\frac{R}{d} \right)^2 \left[1 - \frac{2(1 - (1 + \tau_c)e^{-\tau_c})}{\tau_c^2} \right]$$

where $\tau_c = 2\kappa_\nu R$ is the optical depth through the center of the cloud.

- (b) Find the limiting forms of the flux density for the cases $\tau_c \ll 1$ and $\tau_c \gg 1$. Briefly explain the physical interpretation of these two limits.
- (c) Now, consider the cloud to be a specific HII region with the following properties:

- Radius: $R = 1$ pc
- Distance from Earth: $d = 1$ kpc
- Electron Temperature: $T_e = 10^4$ K
- Uniform electron density: $n_e = 230 \text{ cm}^{-3}$

The radio emission is thermal Bremsstrahlung, for which the absorption coefficient is

$$\kappa_\nu \approx 0.082 \text{ pc}^{-1} \left(\frac{\nu}{\text{GHz}} \right)^{-2.1} \left(\frac{T_e}{\text{K}} \right)^{-1.35} \left(\frac{n_e}{\text{cm}^{-3}} \right)^2$$

Using the general formula from part (a), calculate the flux density in Jansky at two frequencies: 50 MHz and 5 GHz. You may assume the source function is the Planck function, $S_\nu = B_\nu(T_e)$.

- (d) An astronomer measures the flux densities you calculated in part (c) and also measures the source's angular size. From these observational quantities, calculate the “disk-averaged brightness temperature” of the source at both 50 MHz and 5 GHz. Compare your results to the cloud's physical temperature ($T_e = 10^4$ K) and briefly explain why they are different or similar.

5. This problem explores the “on-off source” observation method, a technique used in radio astronomy to determine the physical properties of interstellar gas clouds.

- (a) A cloud is in thermal equilibrium at a physical temperature T . In the Rayleigh-Jeans regime ($h\nu \ll k_B T$), the radiative transfer equation can be expressed in terms of brightness temperature, T_b . Show that this equation takes the form

$$\frac{dT_b}{d\tau_\nu} = -T_b + T$$

Integrate this equation to find the general solution for $T_b(\tau_\nu)$ for an isothermal cloud, assuming an incident brightness temperature of $T_b(0)$ at $\tau_\nu = 0$. What does this solution simplify to when the cloud is optically thick ($\tau_\nu \gg 1$)?

- (b) An observer points their telescope at the cloud where it is backlit by a strong background radio source with a known brightness temperature $T_b(0)$. The total brightness temperature measured through the cloud is $T_b(\tau_\nu)$. The absorption signal, which we will call T_{abs} , is defined as the *decrease* in brightness relative to the background $T_{\text{abs}} = T_b(0) - T_b(\tau_\nu)$. Write an expression for T_{abs} .
- (c) Next, the observer measures the signal under the condition that the background source is absent (i.e., $T_b(0) = 0$). The resulting signal is purely the thermal emission from the cloud, which we will call T_{em} . Write an expression for T_{em} .
- (d) By combining the two measurements, T_{abs} and T_{em} , show that the optical depth and physical temperature of the cloud can be determined as:

$$\tau_\nu = -\ln \left(1 - \frac{T_{\text{abs}} + T_{\text{em}}}{T_b(0)} \right)$$

$$T = T_b(0) \frac{T_{\text{em}}}{T_{\text{abs}} + T_{\text{em}}}$$

Briefly describe an observational strategy that would allow an astronomer to measure T_{abs} and T_{em} in practice.

6. Consider the radiation beam from a pulsar, which is highly collimated. While the true structure of these beams is incredibly complex, we can explore their fundamental properties using a simplified but instructive model. In this problem, let us assume a conical “top-hat” beam, i.e., the specific intensity given by

$$I_\nu(\theta) = I_{\nu 0}, \quad 0 \leq \theta \leq \theta_{\text{max}},$$

$$= 0, \quad \theta > \theta_{\text{max}}.$$

- (a) Make a sketch of the beam pattern, $I_\nu(\theta)$.
- (b) Calculate the specific energy density u_ν and specific radiation pressure P_ν for this beam.
- (c) Find the relation between P_ν and u_ν . What is the limiting value of this relation when $\theta_{\text{max}} \rightarrow 0$ and $\theta_{\text{max}} \rightarrow \pi/2$? Explain the physical significance of these limits.
7. Consider a monochromatic wave propagating primarily along the z -direction. Under the paraxial approximation, the complex amplitude $f(\vec{x}_\perp, z)$ of the wave can be written as

$$f(\vec{x}_\perp, z) = e^{i\omega z/c} \int \frac{d^2 k_\perp}{(2\pi)^2} \tilde{F}(\vec{k}_\perp) \exp \left(-\frac{ic\vec{k}_\perp^2 z}{2\omega} \right) e^{i\vec{k}_\perp \cdot \vec{x}_\perp},$$

where $\vec{x}_\perp = (x, y)$ and $\vec{k}_\perp = (k_x, k_y)$ are the transverse coordinates and wave vectors respectively, while z is coordinate along the propagation direction. The quantity $\tilde{F}(\vec{k}_\perp)$ denote the spatial frequency components.

Suppose we are given the amplitude $f(\vec{x}_{\perp,0}, z_0)$ at some location $\vec{x}_0$, the amplitude in any other location can be written as

$$f(\vec{x}_\perp, z) = \int d^2 x_{\perp,0} f(\vec{x}_{\perp,0}, z_0) G(z - z_0, \vec{x}_\perp - \vec{x}_{\perp,0}).$$

Find the form of the Green's function $G(z - z_0, \vec{x}_\perp - \vec{x}_{\perp,0})$.

8. An astronomer uses a single-dish radio telescope of diameter D operating at an wavelength λ , corresponding to a frequency ν . To a good approximation, the telescope has a Gaussian beam profile $B_\nu(\theta)$ with a half-power beamwidth (HPBW) or full-width at half-maximum (FWHM) given by

$$\theta_{\text{HPBW}} = 1.2 \frac{\lambda}{D}.$$

The profile is normalized such that $B_\nu(\theta = 0) = 1$.

The telescope is pointed at the centre of a distant spherical astronomical source of radius R_s at a distance r having uniform specific intensity with a brightness temperature $T_{b,s}$. The the total flux density F_ν (in units of $\text{W m}^{-2}\text{Hz}^{-1}$) received from the source is related to the specific intensity I_ν (in units of $\text{W m}^{-2}\text{Hz}^{-1}\text{sr}^{-1}$) by

$$F_\nu = \int d\Omega I_\nu(\theta) B_\nu(\theta).$$

We will assume that beam is narrow enough that we can approximate all angles as small, i.e., $\sin \theta \approx \theta$ and $\cos \theta \approx 1$.

- Derive an expression for the flux density F_ν in terms of $T_{b,s}$, R_s , r , θ_{HPBW} .
- One can convert the flux density to a specific intensity by dividing the flux density by the solid angle of the telescope beam given by

$$\Omega_B = \int d\Omega B_\nu(\theta).$$

The specific intensity can then be written in terms of a brightness temperature as

$$T_b = \frac{c^2}{2\nu^2 k_B} \frac{F_\nu}{\Omega_B}.$$

This quantity can be measured by the astronomer.

Write an expression for T_b in terms of the physical parameters of the source and telescope. Discuss under what conditions the estimated brightness temperature T_b will be close to the actual brightness temperature of the source.

- Suppose the telescope has a diameter of $D = 100$ m and is observing at a frequency of $\nu = 5$ GHz. Suppose the source is a cold gas cloud with a physical temperature of $T = 100$ K and radius $R_s = 1$ pc.

What would be the estimated brightness temperature T_b if the cloud is at a distance of $r = 100$ pc, 1 kpc and 10 kpc? Explain the results.

9. Consider the polarization tensor

$$\rho_{ij} = \begin{pmatrix} \langle |E_{0x}|^2 \rangle & \langle E_{0x} E_{0y}^* \rangle \\ \langle E_{0x}^* E_{0y} \rangle & \langle |E_{0y}|^2 \rangle \end{pmatrix},$$

where the time average of the quantities are defined as

$$\langle E_{0i} E_{0j}^* \rangle = \frac{\int_0^T dt E_{0i}(t) E_{0j}^*(t)}{T}.$$

- If we write the tensor in an alternate form as

$$\rho_{ij} = \frac{1}{2} \begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix},$$

then find the expressions for the Stokes parameters I, Q, U, V in terms of the $\langle E_{0i} E_{0j}^* \rangle$'s.

- Show that $\det \rho \geq 0$. You may use the Schwarz inequality without proving it.
- Calculate the two eigenvalues of the polarization tensor ρ_{ij} . Express your answers in terms of the total intensity I and the degree of polarization $P = \sqrt{Q^2 + U^2 + V^2}/I$. Show that the intensity of the fully polarized component of the wave is equal to the difference between the two eigenvalues, and the intensity of the unpolarized component is equal to twice the smaller eigenvalue.

10. A radio telescope is observing a pulsar. The true emission from the pulsar is described by the following intrinsic Stokes parameters

$$I_{\text{int}} = 10 \text{ Jy}, \quad Q_{\text{int}} = 6 \text{ Jy}, \quad U_{\text{int}} = 0 \text{ Jy}, \quad V_{\text{int}} = 8 \text{ Jy}.$$

However, the telescope's receiver system is imperfect. The signal from the y -dipole is measured with an amplitude that is 10% lower than the x -dipole (known as the gain error). Furthermore, a constant phase of $\delta_\epsilon = +30^\circ$ is introduced into the y -channel signal (known as phase error) before it is correlated with the x -channel signal.

- Find the degree of linear, circular and total polarization of the intrinsic pulsar emission.
 - Determine the true complex electric field amplitudes, E_{0x} and E_{0y} , that would produce these Stokes parameters. For simplicity, assume E_{0x} is purely real and positive, and let the total intensity be normalized such that $\langle |E_{0x}|^2 \rangle + \langle |E_{0y}|^2 \rangle = I_{\text{int}}$. For the purpose of this calculation, you can treat the complex amplitudes E_{0x} and E_{0y} as constant values that represent the time-averaged properties of the signal.
 - Find the measured Stokes parameters $I_{\text{meas}}, Q_{\text{meas}}, U_{\text{meas}}, V_{\text{meas}}$ that would be recorded by the telescope, taking into account the gain and phase errors. Hence find the measured degree of linear, circular and total polarization. Did the instrumental errors cause a conversion or "leakage" of power between the linear and circular polarization components? Did the instrumental errors cause a change in the total degree of polarization?
11. An astronomer uses the Giant Metrewave Radio Telescope (GMRT) to observe a distant quasar that emits a signal known to be 100% linearly polarized with an intrinsic position angle of $\psi_{\text{int}} = 0^\circ$. The signal travels through a magnetized plasma cloud in our galaxy. The cloud is estimated to have a path length of $L = 300 \text{ pc}$, an average electron density of $n_e = 0.1 \text{ cm}^{-3}$, and a magnetic field component parallel to the line of sight of $B_{\parallel} = +2.0 \mu\text{G}$.

The observation is conducted in a band centred at 610 MHz with a bandwidth of 32 MHz.

- Calculate the change in polarization angle $\Delta\psi$ (in degrees) due to Faraday rotation at the lower and upper edges of the observing band.
 - Faraday rotation can be described as a transformation of the complex electric field vector. If the initial field components are (E_{0x}, E_{0y}) , the rotated components (E'_{0x}, E'_{0y}) are given by:
$$\begin{aligned} E'_{0x} &= E_{0x} \cos(\Delta\psi) - E_{0y} \sin(\Delta\psi) \\ E'_{0y} &= E_{0x} \sin(\Delta\psi) + E_{0y} \cos(\Delta\psi) \end{aligned}$$
- Using these transformation rules and the fundamental definitions of the Stokes parameters in terms of the complex electric fields, show how the different Stokes parameters transform under Faraday rotation.
- The receiver effectively averages the Stokes parameters over its bandwidth. Find the observed Stokes parameters (I, Q, U, V) after averaging over the bandwidth. Hence find the observed degree of linear polarization.
12. An astronomer is analyzing data from a newly discovered pulsar. They have measured the arrival times of pulses and the position angle (PA) using the Stokes parameters of the linear polarization at several frequencies. Their goal is to use this pulsar as a probe to characterize the interstellar medium along the line of sight.

Frequency (MHz)	Arrival Time (s)	Position Angle (PA) (degrees)
800	t_0 (reference)	20.0
700	$t_0 + 0.10$	120.0
600	$t_0 + 0.25$	40.0
500	$t_0 + 0.51$	140.0

- Using the time-delay vs frequency, calculate the DM in pc cm^{-3} .
- Using the PA vs frequency, determine the RM in rad m^{-2} .

Hint: Plot the PA against λ^2 first. You will notice the points do not form a straight line. This is because polarization angles are only measurable modulo an angle π . You will need to add or subtract multiples of π to the appropriate data points to "unwrap" the phase and reveal the true linear relationship. From the slope of your corrected plot, determine the RM.

- (c) Using your derived values for DM and RM, calculate the average line-of-sight magnetic field strength, $B_{\parallel}$ in μG .
- (d) Suppose another pulsar is discovered in the same part of the sky with a nearly identical, large DM, but its RM is measured to be close to zero. Provide a physical explanation for how this could be possible.
13. Consider an idealized HII region, which appears as a circular disk with an angular diameter of $\theta = 12.8$ arcseconds. The flux density is measured at four radio frequencies

- $F_1 = 0.03$ Jy at $\nu_1 = 200$ MHz
- $F_2 = 0.18$ Jy at $\nu_2 = 500$ MHz
- $F_3 = 0.70$ Jy at $\nu_3 = 1$ GHz
- $F_4 = 0.68$ Jy at $\nu_4 = 5$ GHz

Assume the nebula is a uniform, spherical plasma of pure ionized hydrogen ($Z = 1$) at a single temperature T_e .

- (a) Calculate the radio spectral index α (where $F_{\nu} \propto \nu^{\alpha}$) in the low-frequency part of the spectrum using the data at 200 MHz and 500 MHz. Based on this value, argue whether the emission is more likely to be thermal bremsstrahlung or non-thermal synchrotron self-absorption.
- (b) Calculate the brightness temperature of the nebula at all four frequencies. The brightness temperature is the temperature a blackbody would need to have to produce the observed specific intensity at that frequency. Based on these values, provide a best possible estimate for the physical electron temperature (T_e) of the plasma and justify your reasoning.
- (c) Using your estimated T_e from part (a), calculate the optical depth τ_{ν} at the highest frequency (5 GHz), where the nebula is most likely to be optically thin. From this, determine the Emission Measure (EM) of the nebula in units of pc cm^{-6} . For free-free emission, you may assume the absorption coefficient is given by

$$\kappa_{\nu} = 0.082 \text{ pc}^{-1} \left(\frac{\nu}{\text{GHz}} \right)^{-2.1} \left(\frac{T_e}{\text{K}} \right)^{-1.35} \left(\frac{n_e}{\text{cm}^{-3}} \right)^2.$$

- (d) Suppose other measurements have determined the distance to this HII region is $d = 31$ kpc. Calculate the physical diameter D of the nebula (in parsecs) and determine the average electron number density n_e (in cm^{-3}). For simplicity, you may assume $\text{EM} = n_e^2 D$.
- (e) Estimate the total mass of ionized hydrogen in the nebula, expressing your answer in solar masses ($M_{\odot}$).
- (f) Using the physical parameters you derived, calculate the theoretical turnover frequency ν_{turn} (the frequency at which $\tau_{\nu} = 1$). Does the shape of the observed spectrum (i.e., the relative flux densities at the three frequencies) make sense in light of this turnover frequency? Explain briefly.
14. The Crab Nebula is a supernova remnant with a diameter of about 3.4 parsecs and a magnetic field strength estimated to be $B \approx 3 \times 10^{-4}$ G. It is a powerful synchrotron source, emitting across the electromagnetic spectrum. Assume the radio emission is produced by electrons with a Lorentz factor of $\gamma = 10^4$.
- (a) Calculate the critical frequency (ν_c) in GHz at which an electron with $\gamma = 10^4$ emits most of its synchrotron power. Does this fall in the radio part of the spectrum?
- (b) Calculate the total power (in ergs/s) radiated by a single one of these electrons.
- (c) The synchrotron lifetime of an electron is the timescale on which it loses its energy, defined as $\tau = \mathcal{E}/P_{\text{rad}}$, where $\mathcal{E} = \gamma m_e c^2$. Calculate the synchrotron lifetime (in years) for these $\gamma = 10^4$ electrons.
- (d) The electrons powering the highest-energy emission (X-rays and gamma-rays) have $\gamma \sim 10^9$. What is their synchrotron lifetime? How does this lifetime compare to the age of the Crab Nebula (which exploded in 1054 AD)? Given that the nebula is still a bright X-ray source, what does this imply about these highest-energy electrons?
15. You observe a compact radio source and measure its flux density at several frequencies. The resulting spectrum is a power-law, $S_{\nu} \propto \nu^{-\alpha}$, at high frequencies and turns over at low frequencies, following $S_{\nu} \propto \nu^{5/2}$ due to synchrotron self-absorption.

The observed spectrum has the following properties:

- The spectral index in the optically thin regime is $\alpha = 0.75$.
 - The spectrum peaks at a turnover frequency of $\nu_t = 800$ MHz.
 - The flux density at the turnover frequency is $S_t = 5$ Jy ($1\text{Jy} = 10^{-23} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}$).
 - The source is resolved to have an angular diameter of $\theta = 10$ milliarcseconds.
- (a) What is the power-law index, p , of the energy distribution of the radiating electrons?
- (b) The turnover occurs where the optical depth $\tau_{\nu_t} \approx 1$. Find the brightness temperature T_b of the source at the turnover frequency.
- (c) The brightness temperature of a self-absorbed synchrotron source is related to the energy of the electrons emitting at that frequency. By relating the electron energy $\mathcal{E} \approx k_B T_b$ to the synchrotron critical frequency formula ($\nu_t \propto \gamma^2 B$), derive an estimate for the magnetic field strength B (in Gauss) within the source.