

The Radio Sky: Problem Sheet 0

IUCAA-NCRA Graduate School

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- The questions in this sheet are based on courses you are expected to have completed before attending this course.
- You may look up textbooks and/or consult friends for solving the problems, but make sure you understand the solutions and the associated physics.
- You need *not* submit the solutions. However, if you find any of these questions non-trivial/difficult, please revise the corresponding chapters from relevant textbooks. In the extreme case you are unable to solve a problem, you may seek help from the instructor.

1. **Properties of Maxwell's equations:** The Maxwell's equations are

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0, & \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \\ \vec{\nabla} \cdot \vec{E} &= 4\pi\rho, & \vec{\nabla} \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{j}.\end{aligned}$$

(a) Show that $\vec{E}$ and $\vec{B}$ satisfy the equations

$$\begin{aligned}\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} &= 4\pi \vec{\nabla} \rho + \frac{4\pi}{c^2} \frac{\partial \vec{j}}{\partial t}, \\ \nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} &= -\frac{4\pi}{c} \vec{\nabla} \times \vec{j}.\end{aligned}$$

(b) Starting from the Maxwell's equations, derive the Poynting's theorem

$$\frac{\partial}{\partial t} \left(\frac{E^2 + B^2}{8\pi} \right) + \vec{\nabla} \cdot \left(\frac{c}{4\pi} \vec{E} \times \vec{B} \right) = -\vec{j} \cdot \vec{E}.$$

Interpret all the terms of the above equation.

2. **Larmor's formula:** For a point charge q moving along some arbitrary trajectory $\vec{x}_q(t)$, the electric and magnetic fields are given by

$$\vec{E}(t, \vec{x}) = \left[\frac{q(\hat{n}_R - \dot{\vec{x}}_q/c)(1 - \dot{\vec{x}}_q^2/c^2)}{(1 - \hat{n}_R \cdot \dot{\vec{x}}_q/c)^3 R^2} \right]_{t_r} + \left[\frac{q\hat{n}_R \times \{(\hat{n}_R - \dot{\vec{x}}_q/c) \times \ddot{\vec{x}}_q/c\}}{c(1 - \hat{n}_R \cdot \dot{\vec{x}}_q/c)^3 R} \right]_{t_r},$$

and

$$\vec{B}(t, \vec{x}) = [\hat{n}_R]_{t_r} \times \vec{E}(t, \vec{x}),$$

where $R = |\vec{x} - \vec{x}_q(t_r)|$, $t_r = t - |\vec{x} - \vec{x}_q(t_r)|/c$ is the retarded time, and $\hat{n}_R$ is a unit vector in the direction of $\vec{x} - \vec{x}_q(t_r)$.

(a) Interpret the terms in the above expressions and identify which of the terms correspond to the radiation? Isolate those terms in the expression for the fields and write them as $\vec{E}_{\text{rad}}(t, \vec{x})$ and $\vec{B}_{\text{rad}}(t, \vec{x})$.

- (b) When the particles are non-relativistic $\dot{\vec{x}}_q \ll c$, and $|\vec{x}| \gg |\vec{x}_q|$ (the distance to an object is much larger than its characteristic size), show that the radiation fields reduce to

$$\vec{E}_{\text{rad}}(t, \vec{x}) = \frac{q\hat{n} \times (\hat{n} \times \ddot{\vec{x}}_q(t - r/c))}{c^2 r},$$

$$\vec{B}_{\text{rad}}(t, \vec{x}) = \hat{n} \times \vec{E}_{\text{rad}}(t, \vec{x}),$$

where $r = |\vec{x}|$ and $\hat{n} = \vec{x}/r$.

- (c) Show that the flux associated with this radiation field is given by

$$\vec{F}_{\text{rad}} = \frac{q^2 a_q^2}{4\pi c^3 r^2} \sin^2 \theta \hat{n},$$

where $\vec{a}_q \equiv \ddot{\vec{x}}_q$ and θ is the angle between $\hat{n}$ and $\vec{a}_q$, i.e., between the direction of the radiation and acceleration of the particle.

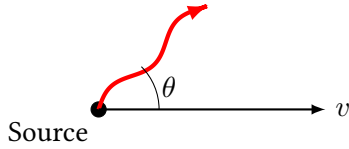
- (d) Show that the total flux integrated over a surface of radius r is

$$P_{\text{rad}} = \frac{2q^2 a_q^2}{3c^3}.$$

3. **Beaming of radiation:** Consider a radiation source moving at velocity v along the $+x$ -axis in the lab frame L . Let the rest frame of the source be denoted as R . The source emits waves with the wave vector

$$\vec{k} = \left(\frac{\omega}{c} \cos \theta, \frac{\omega}{c} \sin \theta, 0 \right)$$

where θ is the propagation angle from the x -axis and ω is the angular frequency.



- (a) Derive the relativistic Doppler effect for frequency

$$\omega_L = \gamma \omega_R \left(1 + \frac{v}{c} \cos \theta_R \right),$$

where θ_R is the propagation angle in the rest frame of the source and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Simplify the expression for $\theta_R = 0$ and $\theta_R = \pi$, and explain the physical meaning.

- (b) Derive the angle transformation

$$\tan \theta_L = \frac{\sin \theta_R}{\gamma(\cos \theta_R + v/c)}.$$

For $\theta_R = \pi/2$, show $\theta_L \approx \gamma^{-1}$ when $v \rightarrow c$. How does this relation lead to beaming of the radiation originating from a relativistic source?

4. **Compton and inverse Compton scattering:** Consider the scattering of a photon off an electron of rest mass m . The incident photon has four-momentum moves in a direction $\hat{k}$ with a frequency ω , while the scattered photon moves along $\hat{n}$ with frequency ω_1 . The initial electron has a velocity $\vec{u}_i$. Let ψ be the angle between $\hat{k}$ and $\hat{n}$, α_k between $\vec{u}_i$ and $\hat{k}$, α_n between $\vec{u}_i$ and $\hat{n}$.

- (a) Using energy and momentum conservation, derive the relation between the incident and scattered frequency of the photon

$$\omega_1 = \omega \frac{1 - \frac{u_i}{c} \cos \alpha_k}{1 - \frac{u_i}{c} \cos \alpha_n + \frac{\hbar \omega}{\gamma_i m c^2} (1 - \cos \psi)},$$

where $\gamma_i = 1/\sqrt{1 - u_i^2/c^2}$. Discuss the limits under which the result approaches that corresponding to Thomson scattering.

- (b) Show that, for an electron initially at rest $u_i = 0$, the expression simplifies to

$$\omega_1 = \frac{\omega}{1 + (\hbar\omega/mc^2)(1 - \cos\psi)}.$$

Verify $\omega_1 < \omega$, explaining energy transfer to the electron (Compton scattering).

- (c) In terms of wavelength $\lambda = 2\pi c/\omega$, derive the Compton shift:

$$\lambda_1 - \lambda = \frac{h}{mc}(1 - \cos\psi) = \lambda_c(1 - \cos\psi),$$

where $\lambda_c = h/mc$ is the Compton wavelength. Compute λ_c for electrons and explain the Thomson limit.

- (d) Consider an ultra-relativistic electron with $u_i \rightarrow c$, $\gamma_i \gg 1$, but $\hbar\omega \ll \gamma_i mc^2$. Argue that the scattering is Thomson-like in the rest frame of the electron.

Given relativistic beaming confines the scattered photon to $\alpha_n \sim \gamma_i^{-1} \approx 0$, show that

$$\frac{\omega_1}{\omega} \approx 2\gamma_i^2 \left(1 - \frac{u_i}{c} \cos\alpha_k\right).$$

Show that the maximum gain in the photon energy occurs for head-on collision ($\alpha_k = \pi$), and is given by $\omega_1 \approx 4\gamma_i^2\omega$ (inverse Compton scattering).

- (e) For electrons with $\gamma_i \sim 10^3$ and radio photons ($\omega \sim 1$ GHz), verify the inverse Compton condition $\hbar\omega \ll \gamma_i mc^2$. For a head-on collision, compute the up-scattered frequency in Hz and identify the electromagnetic spectrum region (e.g., optical, X-ray).

5. **The quantum simple harmonic oscillator:** Consider a quantum mechanical particle in an one-dimensional simple harmonic oscillator potential $V(x) = m\omega^2 x^2/2$.

- (a) Show that the eigenvalues of the Hamiltonian H of the system *cannot* be negative.
(b) Show that the Hamiltonian of the system can be written as

$$H = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right),$$

where the operators $a_{\pm}$ are defined as

$$a_{\pm} = \sqrt{\frac{1}{2\hbar m\omega}} (m\omega x \mp ip).$$

- (c) Show that if $|\psi\rangle$ is an eigenstate of H with eigenvalue E , then $a_{\pm}|\psi\rangle$ too are eigenstates of H but with eigenvalues $E \pm \hbar\omega$.
(d) Argue that the ground state $|\psi_0\rangle$ of the system (i.e., the eigenstate with the lowest value of E) must satisfy

$$a_-|\psi_0\rangle = 0.$$

Solve the equation to obtain the normalized ground state. What is the value of the ground state energy?

- (e) Apply a_+ successively on the ground state and show that the n th energy eigenstate is given by

$$|\psi_n\rangle = \frac{1}{\sqrt{n!}} (a_+)^n |\psi_0\rangle.$$

- (f) Write down the time-independent Schrödinger equation for a particle moving in the potential, solve it using appropriate boundary conditions and obtain the energy levels as well as the wave functions.

6. **The hydrogen atom:** Consider the hydrogen atom which has a potential

$$V(r) = -\frac{e^2}{r}.$$

Because the Coulomb potential is spherically symmetric, the Schrödinger equation can be solved by separation of variables in *spherical* coordinates. The solutions to the angular part are the spherical harmonics. Use the power series method to solve the radial equation. Find the recursion formula for the coefficients, and determine the allowed energies as well as the degeneracy of the bound states.

7. **Time-independent perturbation theory:** Consider the Hamiltonian $H = H_0 + \epsilon H_1$, where $\epsilon \ll 1$, and assume the unperturbed system H_0 has non-degenerate energy levels with known eigenstates $|E_i^{(0)}\rangle$ and eigenvalues $E_i^{(0)}$, forming an orthonormal basis.

(a) Starting from the full eigenvalue equation

$$(H_0 + \epsilon H_1) |E_i\rangle = E_i |E_i\rangle,$$

expand the eigenstate as

$$|E_i\rangle = \sum_{n=0}^{\infty} \epsilon^n |E_i^{(n)}\rangle,$$

and the energy as

$$E_i = \sum_{n=0}^{\infty} \epsilon^n E_i^{(n)}.$$

Substitute these into the equation and equate coefficients of powers of ϵ to derive the equations for the zeroth-order, first-order, and second-order terms (up to the equation for $|E_i^{(2)}\rangle$).

(b) For the first-order energy correction, take the inner product of the ϵ^1 equation with $\langle E_i^{(0)}|$ and show that $E_i^{(1)} = \langle E_i^{(0)}|H_1|E_i^{(0)}\rangle$.

(c) For the first-order wave function correction, take the inner product of the ϵ^1 equation with $\langle E_j^{(0)}|$, where $j \neq i$, and derive

$$\langle E_j^{(0)}|E_i^{(1)}\rangle = \frac{\langle E_j^{(0)}|H_1|E_i^{(0)}\rangle}{E_i^{(0)} - E_j^{(0)}}.$$

Then, expand $|E_i^{(1)}\rangle = \sum_j a_j |E_j^{(0)}\rangle$ and explain how to determine the coefficients a_j for $j \neq i$.

8. **Time-dependent perturbation theory and Fermi's golden rule:** Consider the Hamiltonian $H = H_0 + \epsilon H_1(t)$ with $\epsilon \ll 1$, where the unperturbed eigenstates $|E_i\rangle$ and energies E_i corresponding to H_0 are known and orthonormal.

(a) Expand the wave function as

$$|\psi(t)\rangle = \sum_i c_i(t) |E_i\rangle e^{-iE_i t/\hbar}.$$

Substitute into the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle,$$

and derive the coupled differential equations

$$\dot{c}_f(t) = -\frac{i}{\hbar} \epsilon \sum_i c_i(t) \langle E_f|H_1(t)|E_i\rangle e^{i\omega_{fi}t},$$

where $\omega_{fi} = (E_f - E_i)/\hbar$.

(b) Assume ϵ is small and expand

$$c_i(t) = c_i^{(0)} + \epsilon c_i^{(1)} + \dots$$

Derive the first-order equation

$$\dot{c}_f^{(1)} = -\frac{i}{\hbar} \sum_i c_i^{(0)} \langle E_f | H_1 | E_i \rangle e^{i\omega_{fi}t}.$$

For a system starting in state $|E_a\rangle$ at $t = 0$ (so $c_i(0) = \delta_{ia}$), integrate to find for $f \neq a$:

$$c_f^{(1)}(t) = -\frac{i}{\hbar} \int_0^t dt' \langle E_f | H_1(t') | E_a \rangle e^{i\omega_{fa}t'}.$$

(c) For a periodic perturbation $\epsilon H_1(t) = H^{\text{trans}} e^{-i\omega t}$ (with H^{trans} time-independent), derive the transition amplitude

$$c_f(t) = -\frac{i}{\hbar} \langle E_f | H^{\text{trans}} | E_a \rangle e^{i(\omega_{fa}-\omega)t/2} \frac{\sin[(\omega_{fa}-\omega)t/2]}{(\omega_{fa}-\omega)/2}, \quad f \neq a.$$

(d) Compute the transition probability $P_{a \rightarrow f}(t) = |c_f(t)|^2$ and describe its behaviour (e.g., peak at resonance $\omega = \omega_{fa}$, width $\propto 1/t$).

(e) Derive the transition rate

$$R_{a \rightarrow f}(t) = \frac{dP_{a \rightarrow f}(t)}{dt}.$$

In the long-time limit $t \gg (\omega_{fa} - \omega)^{-1}$, show that the rate simplifies to

$$\lim_{t \rightarrow \infty} R_{a \rightarrow f}(t) = \frac{2\pi}{\hbar^2} |\langle E_f | H^{\text{trans}} | E_a \rangle|^2 \delta_D(\omega_{fa} - \omega).$$

This is known as the Fermi's golden rule. Explain the energy conservation implied (transitions only for $E_f > E_a$).

(f) For the case $\epsilon H_1(t) = H^{\text{trans}} e^{i\omega t}$, derive the analogous golden rule and discuss the transition direction ($E_f < E_a$).

9. **Eigenvalues of angular momentum:** Consider the fundamental commutation relations for a *general* angular momentum operator $\vec{S}$:

$$[S_i, S_j] = i\hbar \sum_k \varepsilon_{ijk} S_k,$$

where S_i , $i = 1, 2, 3$ are the Cartesian components of $\vec{S}$, and ε_{ijk} is the Levi-Civita symbol. Let $|s, m\rangle$ be a simultaneous eigenstate of $S^2 \equiv S_x^2 + S_y^2 + S_z^2$ and S_z with eigenvalues as $\hbar^2 s(s+1)$ and $\hbar m$, respectively.

(a) Show that for a given s , the allowed values of m are $-s, -s+1, \dots, +s$.

(b) Also show that the quantum number s must be an integer or a half-integer.

10. **Spin-1/2 particles and the Pauli matrices:** Consider a spin-1/2 particle (e.g., an electron). Suppose we are working in a basis where S_z is diagonal.

(a) In the above basis, explicitly construct the matrices for S_x , S_y , and S_z .

(b) The Pauli matrices are defined as $\vec{\sigma} = 2\vec{S}/\hbar$. Write down the matrices σ_x , σ_y , and σ_z .

(c) Prove the following identities for the Pauli matrices:

- $[\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k$,
- $\{\sigma_i, \sigma_j\} = 2 \delta_{ij} 1_2$ where 1_2 is the 2×2 identity matrix,
- $\sigma_i \sigma_j = \delta_{ij} 1_2 + i \sum_k \varepsilon_{ijk} \sigma_k$,
- $(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = (\vec{a} \cdot \vec{b}) 1_2 + i \vec{\sigma} \cdot (\vec{a} \times \vec{b})$, where $\vec{\sigma} = \sigma_1 \hat{x} + \sigma_2 \hat{y} + \sigma_3 \hat{z}$ and $\vec{a}, \vec{b}$ are ordinary vectors.

(d) Consider the spin component $\vec{S} \cdot \hat{n}$ in an arbitrary direction given by the angles (θ, ϕ) , where $\hat{n}$ is the unit vector along (θ, ϕ) . Find the eigenstates $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ of this spin component. What happens to the eigenstates when $\theta \rightarrow \theta + 2\pi$?

11. **Combining two spin-1/2 particles:** Consider a system of two spin-1/2 particles (e.g., the electron and proton in a hydrogen atom). The individual spin operators are $\vec{S}_1$ and $\vec{S}_2$, and the total spin is $\vec{S} = \vec{S}_1 + \vec{S}_2$. The basis states for the individual spins are $|s_1, m_1\rangle$ and $|s_2, m_2\rangle$. The combined system can be described in the uncoupled basis $|m_1; m_2\rangle \equiv |s_1 = 1/2, m_1\rangle |s_2 = 1/2, m_2\rangle$.

(a) There are four uncoupled basis states $|m_1; m_2\rangle$

$$\left|+\frac{1}{2}; +\frac{1}{2}\right\rangle, \left|+\frac{1}{2}; -\frac{1}{2}\right\rangle, \left|-\frac{1}{2}; +\frac{1}{2}\right\rangle, \left|-\frac{1}{2}; -\frac{1}{2}\right\rangle \equiv |++\rangle, |+-\rangle, |-+\rangle, |--\rangle.$$

Show that these are all eigenstates of the total z -component operator $S_z = S_{1z} + S_{2z}$ and find their corresponding eigenvalues.

- (b) The total spin-squared operator is $S^2 = (\vec{S}_1 + \vec{S}_2)^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$. Show how the term $2\vec{S}_1 \cdot \vec{S}_2$ can be written in terms of ladder operators $S_{\pm} \equiv S_x \pm iS_y$ as $2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$.
- (c) By operating with S^2 on the four uncoupled basis states, show that $|++\rangle$ and $|--\rangle$ are already eigenstates of S^2 . What is the value of the total spin quantum number s for these states?
- (d) Find the two linear combinations of $|+-\rangle$ and $|-+\rangle$ that are eigenstates of S^2 . What are their corresponding eigenvalues and the value of s ?
- (e) These four resulting states, i.e., the ones found in the previous two steps, are the “triplet” and “singlet” states. Write them down in the form $|s, m\rangle$.

12. **Adding orbital and spin angular momentum ($\vec{L} + \vec{S}$):** Consider an electron with orbital angular momentum $\vec{L}$ (quantum number l) and spin $\vec{S}$ (quantum number $s = 1/2$). The total angular momentum is $\vec{J} = \vec{L} + \vec{S}$. The coupled basis states are denoted by $|j, m\rangle$, while the uncoupled basis is $|l, m_l; s, m_s\rangle$.

- (a) What are the possible values for the total angular momentum quantum number j for a given $l > 0$?
- (b) The coupled states $|j, m\rangle$ can be written as a linear combination of the uncoupled states using Clebsch-Gordan coefficients. For a given m , we can write:

$$|j, m\rangle = C_1 \left|l, m - \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right\rangle + C_2 \left|l, m + \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right\rangle.$$

Derive the Clebsch-Gordan coefficients (C_1 and C_2) for the states with $j = l + 1/2$ and $j = l - 1/2$.