Quantum Mechanics: Take-home Assignment 3

IUCAA-NCRA Graduate School

August - September 2016

08 September 2016

To be returned in NCRA Office 236 on 15 September 2016 (between 10:00 – 11:00)

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

---

1. **Landau levels:** Consider a free electron (ignoring spin) of charge $-e$ moving in a uniform time-independent magnetic field $\boldsymbol{B}$.

   (a) Show that the vector potential
   $$\boldsymbol{A} = \frac{1}{2}\left(\boldsymbol{B} \times \boldsymbol{x}\right)$$
   produces the correct uniform magnetic field $\boldsymbol{B}$.

   (b) Now orient the axes such that the magnetic field is in the z-direction, i.e., $\boldsymbol{B} = B\hat{z}$. Calculate the vector potential for this case.

   (c) Use the gauge variance of $\boldsymbol{B}$ under the transformation $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\varphi$ to show that the quantity
   $$\boldsymbol{A} = -B\ y\ \hat{x}$$
   too is an appropriate vector potential. What φ did you choose to obtain the new vector potential from the old one (give your answer up to an additive constant)?

   (d) Show that the Hamiltonian for this system can be written as
   $$H = \frac{1}{2m_e}\left(P_x - \frac{e}{c}By\right)^2 + \frac{1}{2m_e}(P_y^2 + P_z^2).$$

   (e) Show that $[H, P_x] = [H, P_z] = 0$.

   (f) Given the above commutation relations, we can choose the wave function $\psi(\boldsymbol{x})$ to be a simultaneous eigenfunction of H, P_x, P_z. Let $\psi_{p_x}(x)$ be the eigenfunction of P_x with eigenvalue p_x. Obtain the explicit form of $\psi_{p_x}(x)$. Similarly, write down the explicit form of the eigenfunction $\psi_{p_z}(z)$ of P_z. What are the allowed ranges of the eigenvalues p_x and p_z?

   (g) Try a solution of the form
   $$\psi(\boldsymbol{x}) = \psi_{p_x}(x)\ \psi_{p_z}(z)\ \xi(y),$$
   and show that the Schrödinger equation $H\psi = E\psi$ reduces to an equation which resembles a simple harmonic oscillator system in the y-direction. Hence show that the energy eigenvalues are given by
   $$E_n = (2n+1)\hbar\omega_L + \frac{p_z^2}{2m_e},\ n = 0, 1, 2, \ldots,$$
   where
   $$\omega_L = \frac{eB}{2m_e c}$$
   is the Larmor angular frequency.

[3+2+3+4+2+3+6]

2. **Zeeman effect:** Let a hydrogen atom be placed in a uniform magnetic field $\boldsymbol{B}$.

(a) Assuming the charge of the electron to be $-e$, show that the non-relativistic Schrödinger equation for the system (assuming Coulomb gauge) is

$$\left[-\frac{\hbar^2}{2m_e}\boldsymbol{\nabla}^2 - \frac{e^2}{r} - \frac{i\hbar e}{m_e c}\boldsymbol{A}\cdot\boldsymbol{\nabla} + \frac{e^2}{2m_e c^2}\boldsymbol{A}^2 - \boldsymbol{\mu}\cdot\boldsymbol{B}\right]\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x})$$

(b) Show that the linear term in $\boldsymbol{A}$ can be written as

$$\frac{e}{2m_e c}\boldsymbol{B}\cdot\boldsymbol{L}.$$

(c) Estimate the order of magnitude of the magnetic field (in Gauss) that is required to make the quadratic term in $\boldsymbol{A}$ comparable to the linear term. How does this compare with the typical magnetic fields observed in the Milky Way?

(d) Show that, ignoring the quadratic term but including the spin-orbit coupling term, the Schrödinger equation reduces to

$$\left[-\frac{\hbar^2}{2m_e}\boldsymbol{\nabla}^2 - \frac{e^2}{r} + \xi(r)\boldsymbol{L}\cdot\boldsymbol{S} + \frac{e}{2m_e c}\boldsymbol{B}\cdot(\boldsymbol{L}+2\boldsymbol{S})\right]\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x}),$$

where symbols are identical to those used in the class.

(e) Assume the magnetic field to be "weak" so that the term involving $\boldsymbol{B}$ can be taken as a perturbation over the unperturbed Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m_e}\boldsymbol{\nabla}^2 - \frac{e^2}{r} + \xi(r)\boldsymbol{L}\cdot\boldsymbol{S}.$$

Also orient the axes such that the magnetic field is along the z-direction. Show that the first order corrections to the energy levels are given by

$$\Delta E_{j,m,l} = \left\langle j,m,l\left|\frac{e}{2m_e c}\boldsymbol{B}\cdot(\boldsymbol{L}+2\boldsymbol{S})\right|j,m,l\right\rangle = \frac{eB}{2m_e c}\left[\hbar m + \langle j,m,l\,|\,S_z\,|\,j,m,l\rangle\right].$$

Use the relations derived in the class

$$\begin{aligned}\left|j = l \pm \frac{1}{2}, m, l\right\rangle &= \pm\sqrt{\frac{l+1/2\pm m}{2l+1}}\left|l, m_l = m - \frac{1}{2}, m_s = \frac{1}{2}\right\rangle \\ &\quad + \sqrt{\frac{l+1/2\mp m}{2l+1}}\left|l, m_l = m + \frac{1}{2}, m_s = -\frac{1}{2}\right\rangle,\end{aligned}$$

to show

$$\left\langle j = l \pm \frac{1}{2}, m, l\left|S_z\right|j = l \pm \frac{1}{2}, m, l\right\rangle = \pm\frac{\hbar m}{2l+1}.$$

Hence show that the energy levels are split into

$$\Delta E_{j,m,l} = \hbar\omega_L\ m\left[1 \pm \frac{1}{2l+1}\right],$$

where

$$\omega_L = \frac{eB}{2m_e c}$$

is the Larmor angular frequency.

Note: In the above derivation, the matrix involving S_z is not diagonal and hence the first order perturbation calculation we did above is not valid. However, since $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ is conserved, one can think of $\boldsymbol{S}$ (and also $\boldsymbol{L}$) to be "precessing" about $\boldsymbol{S}$. If we take the time average of the vector $\boldsymbol{S}$, only its projection along $\boldsymbol{J}$ survives, thus validating the above calculation.

(f) If we take the magnetic field to be "strong", then we can ignore the spin-orbit coupling term and treat the magnetic field term as perturbation over the unperturbed Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m_e}\boldsymbol{\nabla}^2 - \frac{e^2}{r}.$$

Show that in this case, the energy corrections are given by

$$\Delta E_{l,m_l,m_s} = \hbar\omega_L \ (m_l + 2m_s).$$

[2+2+3+2+7+3]

3. **Fine-structure of the Balmer series lines:** In the class, we saw that the different energy levels of the hydrogen atom are split because of the relativistic corrections. For example, the Lyman series lines are split into a doublet

$$1S_{1/2} \longleftrightarrow nP_{1/2}, \ 1S_{1/2} \longleftrightarrow nP_{3/2}.$$

Consider the Balmer series lines which connect the $n = 2$ state with higher states $n > 2$.

(a) Write down all the possible *allowed* transitions (accounting for the fine-structure) between different levels of $n = 2$ and any $n > 2$ states. Please give short justifications (based on selection rules) for each of these transitions.

(b) What is the number of *distinct* components of the Balmer line for a $2 \longleftrightarrow n$ transition ($n > 2$)?

[3+2]

4. **Isotope shift:** Let us assume that the nucleus of an hydrogen-like atom, instead of being a point charge Ze, is actually a uniformly charged sphere of radius $R = r_0 A^{1/3}$, where A is the mass number of the nucleus and $r_0 \approx 10^{-15}$ m.

(a) Show that the electrostatic potential of the nucleus is given by

$$V(r) = \begin{cases} \dfrac{Ze^2}{2R}\left(\dfrac{r^2}{R^2} - 3\right) & r < R, \\ -\dfrac{Ze^2}{r} & r \geq R \end{cases}$$

(b) If we write the quantum Hamiltonian operator as

$$H = H_0 + H_1,$$

with

$$H_0 = -\frac{\hbar^2}{2\mu_e}\boldsymbol{\nabla}^2 - \frac{Ze^2}{r},$$

what is the form of H_1?

(c) Treating H_1 as a small perturbation over H_0, show that the first-order energy shift is given by

$$\Delta E = \frac{Ze^2}{2R}\int_0^R \mathrm{d}r \ r^2 \ |R_{nl}(r)|^2 \left(\frac{r^2}{R^2} + \frac{2R}{r} - 3\right),$$

where $R_{nl}(r)$ is the radial part of the wave function.

(d) Argue that R is small enough so that we can approximate $R_{nl}(r) \approx R_{nl}(0)$ in the above integral. Hence calculate ΔE. You should look up the forms of the hydrogenic wave functions from any text book or other resource.

(e) Let δE be the energy difference between two isotopes whose charge distributions have radii R and $R + \delta R$, respectively. Show that to the first order in δR, the energy difference is

$$\delta E \approx \frac{4e^2}{5}R^2\frac{Z^4}{a_\mu^3 n^3}\frac{\delta R}{R},$$

where

$$a_\mu = \frac{\hbar^2}{\mu_e e^2}.$$

[4+1+2+5+1]