

**Quantum Mechanics: Take-home Assignment 2**  
**IUCAA-NCRA Graduate School**  
**August - September 2016**

**23 August 2016**

**To be returned in the class on 01 September 2016**

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

1. **Stark effect:** Let a hydrogen atom be placed in a uniform electric field pointing in the  $z$ -direction

$$\mathbf{E}_{\text{ext}} = E_{\text{ext}} \hat{z}.$$

Let us assume that the field  $E_{\text{ext}}$  is small enough so that we can treat the problem perturbatively.

- (a) Assuming the charge of the electron to be  $-e$ , write down the total Hamiltonian of the system.
- (b) Calculate the shift in the ground state  $n = 1$  energy of the atom to the first order in perturbation. You can directly use the explicit forms of the wave functions from any text book.
- (c) Calculate the first order energy shift in the first excited state  $n = 2$ . Remember that the state is degenerate.

[2+3+10]

2. **Transition probabilities:** Consider a time-dependent perturbation potential given by

$$\begin{aligned} H_1(t) &= 0 && \text{for } t < 0, \\ &= H^{\text{trans}} && \text{for } 0 \leq t \leq t_0, \\ &= 0 && \text{for } t > t_0, \end{aligned}$$

where  $H^{\text{trans}}$  is constant. Let us assume  $H^{\text{trans}} \ll H_0$ , the unperturbed potential.

- (a) Assuming that the unperturbed system is in one of its energy eigenstates  $|E_a\rangle$ , find the first order transition probability  $P_{a \rightarrow b}^{\text{trans}}(t)$  from state  $|E_a\rangle$  to state  $|E_b\rangle$  ( $b \neq a$ ) as a function of time. You can start with the expression for first order transition amplitude  $c_b^{(1)}(t)$  for a time-dependent perturbation.
- (b) Plot the quantity

$$f \equiv P_{a \rightarrow b}^{\text{trans}}(t) \times \frac{\hbar^2}{|\langle E_b | H^{\text{trans}} | E_a \rangle|^2}$$

as a function of  $\omega_{ba} \equiv (E_b - E_a)/\hbar$  for  $t = t_0$ . Plot for three values of  $t = 0.5, 1.0, 2.0$  in range  $-10 \leq \omega_{ba} \leq 10$ . All the three cases should be on the same plot, and you should use a computer plotting routine.

[5+5]

3. **Sum Rule:** Show that the sum  $\sum_f f_{fi}$  over all states  $f$  is unity for a given  $i$ , where  $f_{fi}$  are the oscillator strengths. Note that oscillator strengths for absorption ( $E_f > E_i$ ) are positive, while those for emission ( $E_f < E_i$ ) are negative.

[10]

4. **Interaction of matter and the classical radiation field:** Instead of the treating the electromagnetic field quantum mechanically, let us work out the transition rates when the field is classical.

- (a) In the Coulomb gauge  $\nabla \cdot \mathbf{A} = 0$ , the vector potential  $\mathbf{A}$  satisfies the wave equation, and hence can be expanded as

$$\mathbf{A}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \left[ a(\mathbf{k}) \hat{\epsilon}(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} + \text{c.c.} \right], \quad \omega = k c,$$

where  $\hat{\epsilon}(\mathbf{k})$  represents the direction of the polarization. Show that the above expression indeed satisfies the wave equation, and also show that  $\mathbf{k} \cdot \hat{\epsilon}(\mathbf{k}) = 0$ .

- (b) Calculate the energy density

$$u = \frac{1}{8\pi V} \int d^3x [\mathbf{E}^2(\mathbf{x}, t) + \mathbf{B}^2(\mathbf{x}, t)]$$

in the radiation field and show that the specific intensity is given by

$$I_\omega(\hat{n}) = \frac{1}{V} \left( \frac{\omega}{2\pi c} \right)^4 |a(\mathbf{k})|^2.$$

- (c) The interaction between an atom and the radiation field can be described by the Hamiltonian

$$H_1 = \frac{e}{m_e c} \mathbf{A} \cdot \mathbf{P} = -\frac{ie\hbar}{m_e c} \mathbf{A} \cdot \nabla.$$

Show that  $H_1$  can be written as

$$H_1 = V \int \frac{d^3k}{(2\pi)^3} [H^{\text{abs}}(\mathbf{k}) e^{-i\omega t} + H^{\text{emi}}(\mathbf{k}) e^{i\omega t}].$$

Write down the expressions for  $H^{\text{abs}}(\mathbf{k})$  and  $H^{\text{emi}}(\mathbf{k})$ .

- (d) Using the expressions for Fermi's golden rule derived in the class

$$\begin{aligned} R_{a \rightarrow b}^{\text{abs}} &= \frac{2\pi}{\hbar^2} |\langle E_b | H^{\text{abs}} | E_a \rangle|^2 \delta_D(\omega - \omega_{ba}), \\ R_{b \rightarrow a}^{\text{emi}} &= \frac{2\pi}{\hbar^2} |\langle E_a | H^{\text{emi}} | E_b \rangle|^2 \delta_D(\omega - \omega_{ba}), \end{aligned}$$

show that the absorption rate for the radiation-matter interaction is

$$R_{a \rightarrow b}^{\text{abs}} = \frac{4\pi^2 e^2}{m_e^2 c \omega_{ba}^2} \int d\Omega I_\omega(\hat{n}) |\mathcal{M}_{ba}(\mathbf{k})|^2,$$

where

$$\mathcal{M}_{ba}(\mathbf{k}) = \int d^3x \psi_b^*(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}} \hat{\epsilon}(\mathbf{k}) \cdot \nabla \psi_a(\mathbf{x}).$$

- (e) Find out the corresponding emission rate and show that  $R_{b \rightarrow a}^{\text{emi}} = R_{a \rightarrow b}^{\text{abs}}$   
 (f) What is main difference you see between the classical and quantum treatments of the radiation field? What is the explanation for this difference?

[2+6+3+6+5+3]