Methods of Mathematical Physics I: Exercise Sheet 3
IUCAA-NCRA Graduate School
August - September 2014

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1. **Equation of radiative transfer:** The transfer of energy through electromagnetic radiation is described the equation of radiative transfer. The equation describes the evolution of a quantity called **specific intensity** $I_\nu(t, \boldsymbol{x}, \hat{\boldsymbol{n}})$ and is given by

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} I_\nu = j_\nu - \kappa_\nu \; I_\nu$$

where $I_\nu(t, \boldsymbol{x}, \hat{\boldsymbol{n}}) \; \mathrm{d}A \; \mathrm{d}t \; \mathrm{d}\nu \; \mathrm{d}\Omega$ is the energy flowing across an area $\mathrm{d}A$ located at $\boldsymbol{x}$ in the time interval $(t, t + \mathrm{d}t)$ in the solid angle $\mathrm{d}\Omega$ about the direction $\hat{\boldsymbol{n}}$ in the frequency interval $(\nu, \nu + \mathrm{d}\nu)$. The quantity j_ν is the **spontaneous emission coefficient** and κ_ν is the **absorption coefficient**.

(i) Solve the equation (i.e., convert into a set of eight ordinary differential equations) using the method of characetristics. What is the physical significance of the characetristic curves and the parameter s?

(ii) Show that the formal solution of the equation can be written as

$$\begin{aligned} \boldsymbol{x}(t) &= c(t - t_0)\hat{\boldsymbol{n}} \\ I_\nu(t) &= I_\nu(t_0) \exp\left(-c \int_{t_0}^{t} \mathrm{d}t' \; \kappa_\nu(t')\right) + c \int_{t_0}^{t} \mathrm{d}t' \; j_\nu(t') \exp\left(-c \int_{t'}^{t} \mathrm{d}t'' \; \kappa_\nu(t'')\right) \end{aligned}$$

Give a physical interpretation of the solution.

2. **Traffic flow problem:** Consider the idealized flow of traffic along one-dimension (e.g., a one-lane highway). Let $\rho(t, x)$ be the traffic density at a point (t, x) and $v(t, x)$ be the velocity of cars at that point.

(i) Show that if the number of cars is conserved (i.e., no sideway exits), then ρ and v obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$

(ii) Assume that the velocity depends on the density through a relation

$$v(\rho) = c\left(1 - \frac{\rho}{\rho_m}\right)$$

where c is the maximum velocity and ρ_m represents the density during a traffic jam (when $v = 0$). Show that the original differential equation, in terms of suitably re-scaled variables, becomes:

$$\frac{\partial u}{\partial t} + (1 - 2u)\frac{\partial u}{\partial x} = 0$$

(iii) Write down the equations for characteristic curves for this equation. If the initial conditions are given by $x(t = 0) = x_0, \; u(t = 0, x) = f(x)$, then show that the solution is given by

$$x = [1 - 2f(x_0)] \; t + x_0; \quad u = f(x_0)$$

Also note that $\mathrm{d}x/\mathrm{d}t \neq v/c$, i.e., the characteristic velocities do *not* represent the traffic velocity. Rather $\mathrm{d}x/\mathrm{d}t$ is the local speed of the "traffic wave".

(iv) Now consider the initial condition on the traffic density

$$f(x) = \begin{cases} 1 & \text{for } x \leq 0 \\ 0 & \text{for } x > 0 \end{cases}$$

which corresponds to traffic standing at a red light which turns into green at $t = 0$. Show that the solution for the problem is given by

$$x(t) = \begin{cases} x_0 - t & \text{for } x_0 \leq 0 \\ x_0 + t & \text{for } x_0 > 0 \end{cases}$$

and

$$u(t, x) = \begin{cases} 1 & \text{for } x \leq -t \\ 0 & \text{for } x > t \end{cases}$$

Can you interpret the result?

3. **Riemann invariants:** The equations of gas dynamics can be expressed as three conservation equations representing conservation of mass, momentum and energy respectively:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} &= 0 \\ \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + p)}{\partial x} &= 0 \\ \frac{\partial E}{\partial t} + \frac{\partial[(E+p)v]}{\partial x} &= 0 \end{aligned}$$

where $E = \rho\, e + \rho v^2/2$ is the total energy per unit volume, with e being the internal energy per unit mass for the fluid. We have assumed the flow to be in one dimension.

(i) Show that the equations can be simplified to give

$$\begin{aligned} \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \\ \frac{\partial e}{\partial t} + v\frac{\partial e}{\partial x} + \frac{p}{\rho}\frac{\partial v}{\partial x} &= 0 \end{aligned}$$

(ii) The third equation can be written in a simpler form in terms of the entropy s. Using the thermodynamic relation

$$\mathrm{d}e = T\mathrm{d}s + \frac{p}{\rho^2}\mathrm{d}\rho$$

show that the equation becomes

$$\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x} = 0$$

Further, assume an equation of state of the form $p = p(\rho, s)$ and thus eliminate the derivative of p from the second equation (i.e., the momentum conservation equation) in terms of two quantities

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s ; \quad \sigma = \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho$$

The quantity c_s is the local sound speed while σ is a parameter related to the thermal expansivity of the gas.

(iii) Show that the three equations can be written in a compact notation involving matrices

$$\frac{\partial \mathsf{m}}{\partial t} + \mathsf{A}\cdot\frac{\partial \mathsf{m}}{\partial x} = 0$$

where

$$\mathsf{m} = \begin{pmatrix} \rho \\ v \\ s \end{pmatrix}$$

In particular, show that

$$\mathsf{A} = \begin{pmatrix} v & \rho & 0 \\ c_s^2/\rho & v & \sigma \\ 0 & 0 & v \end{pmatrix}$$

(iv) If the matrix A is diagonalizable, it can be written in the form

$$\mathsf{A} = \mathsf{P}\cdot\mathsf{\Lambda}\cdot\mathsf{P}^{-1}$$

Find the matrices P and $\mathsf{\Lambda}$. (This involves finding the eigenvalues and eigenvectors of A.)

(v) Now apply P^{-1} from the left to the set of equations written in matrix form and write down the three resultant equations in explicit form.

(vi) Show that one of the equations can be solved using the method of characteristics:

$$s = \text{ const along the characteristics } \frac{\mathrm{d}x}{\mathrm{d}t} = v$$

(vii) For the other two equations, assume the gas to be isentropic, i.e., $s = $ const. Also assume a polytropic equation of state of the form $p \propto \rho^\gamma$. Write down the two equations using these two simplifications. Show that the two equations are nothing but those written in terms of the Riemann invariants introduced in the class.