Methods of Mathematical Physics I: Exercise Sheet 1

IUCAA-NCRA Graduate School

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1. **Riemann-zeta function:** (i) Expand the function

$$f(x) = x^2, \qquad -\pi < x < \pi$$

in a Fourier series and show that

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$$

(ii) Put $x = \pi$ and show that the value of the zeta function $\zeta(2)$ is given by

$$\zeta(2) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

2. **Triangular wave:** A triangular wave is represented by

$$f(x) = \begin{cases} x & \text{for} \quad 0 < x < \pi \\ -x & \text{for} \quad -\pi < x < 0 \end{cases}$$

Represent $f(x)$ by a Fourier series.

3. **Fourier coefficients using minimization techniques:** A function $f(x)$ (assumed to be quadratically integrable) is to be represented by a *finite* Fourier series. A convenient measure of the accuracy of the series is given by the integrated square of the deviation,

$$\Delta_N = \int_a^b \mathrm{d}x \left[f(x) - \sum_{n=-N}^{N} f_n \, \mathrm{e}^{2\pi \mathrm{i} n x/L} \right]^2, \quad b - a = L$$

Show that the requirement that Δ_N be minimized, i.e.,

$$\frac{\partial \Delta_N}{\partial f_n} = 0,$$

for all n, leads to choosing f_n as given in standard formulae for the Fourier series.

4. **Fourier Transform of a Bessel function:** Use the integral representation of the Bessel function

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \; \mathrm{e}^{\mathrm{i}x\cos\theta}$$

to show that its Fourier transform can be expressed as

$$\tilde{g}(k) = \int_{-\infty}^{\infty} \mathrm{d}x \; J_0(x) \; \mathrm{e}^{-\mathrm{i}kx} = \int_0^{2\pi} \mathrm{d}\theta \; \delta(k - \cos\theta)$$

Noting that the delta function is never satisfied for $|k| > 1$, and that there are two values of the θ which satisfy it for $|k| < 1$, show that

$$\tilde{g}(k) = \begin{cases} \dfrac{2}{\sqrt{1-k^2}} & \text{for} \quad |k| < 1 \\ 0 & \text{for} \quad |k| > 1 \end{cases}$$

5. **Fourier transform for even/odd functions:** (i) Suppose the function $f(x)$ is even. Then show that the Fourier transform is given by the cosine transform

$$\tilde{f}(k) = 2\int_0^{\infty} \mathrm{d}x \; f(x) \; \cos kx$$

What is the inverse relation?

(ii) Repeat the above problem for the case when $f(x)$ is an odd function.

6. **Properties of Fourier transform:** (i) Show that $\tilde{f}(-k) = \tilde{f}^*(k)$ is a necessary and sufficient condition for $f(x)$ to be real.

   (ii) Show that $\tilde{f}(-k) = -\tilde{f}^*(k)$ is a necessary and sufficient condition for $f(x)$ to be pure imaginary.

7. **Fourier transform of an exponential function:** (i) Calculate the Fourier transform of

$$f(t) = \mathrm{e}^{-a|t|}; \qquad a \geq 0$$

   (ii) Calculate the Fourier transform of

$$g(t) = \begin{cases} \mathrm{e}^{-at} & \text{for} \quad t > 0 \\ -\mathrm{e}^{at} & \text{for} \quad t < 0 \end{cases}$$

   where $a \geq 0$ as before.

   (iii) Using the above results, show that

$$\int_0^\infty \mathrm{d}\omega \, \frac{\cos \omega x}{\omega^2 + a^2} = \frac{\pi}{2a} \, \mathrm{e}^{-ax}, \qquad \int_0^\infty \mathrm{d}\omega \, \frac{\omega \, \sin \omega x}{\omega^2 + a^2} = \frac{\pi}{2} \, \mathrm{e}^{-ax}, \qquad x > 0$$

   (iv) Now consider another function

$$h(t) = \Theta(t) - \Theta(-t) = \begin{cases} 1 & \text{for} \quad t > 0 \\ -1 & \text{for} \quad t < 0 \end{cases}$$

   Show that $\lim_{a \to 0} g(t) = h(t)$, where $g(t)$ is defined in (ii). Then calculate the Fourier transform $\tilde{h}(\omega)$.

   (v) Show that we can express the step function as

$$\Theta(t) = \frac{1}{2}[h(t) + 1]$$

   What is the Fourier transform of $\Theta(t)$?

8. **Some symmetry properties of Fourier transform:** If $\tilde{f}(\omega)$ is the Fourier transform of $f(t)$, then show that

   (i) the Fourier transform of $f(at)$ is $\tilde{f}(\omega/a)/|a|$

   (ii) the Fourier transform of $f(t - t_0)$ is $\mathrm{e}^{\mathrm{i}\omega t_0} \tilde{f}(\omega)$

9. **Repeated application of Fourier transform operator:** Let the Fourier transform operator be defined as

$$\left(\hat{F} f\right)(x) = \int \mathrm{d}y \; \mathrm{e}^{-\mathrm{i}xy} \; f(y).$$

   Show that the operator $(2\pi)^{-1} \hat{F}^2$ is the parity operator.

10. **Dirac delta function:** Verify that the function

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi - \phi')} \qquad (0 \leq \phi, \phi' \leq 2\pi)$$

   is a Dirac delta function by showing that it satisfies the definition of a Dirac delta function:

$$\int_0^{2\pi} \mathrm{d}\phi \; f(\phi) \; \left[\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi - \phi')} \right] = f(\phi')$$

   Hint: Represent $f(\phi)$ by an exponential Fourier series.

11. **Linear quantum oscillator:** A linear quantum oscillator in its ground state has a wave function

$$\psi(x) = a^{-1/2} \pi^{-1/4} \; \mathrm{e}^{-x^2/2a^2}$$

   Show that the corresponding momentum function is

$$\tilde{\psi}(p) = a^{1/2} \pi^{-1/4} \hbar^{-1/2} \; \mathrm{e}^{-a^2 p^2/2\hbar^2}$$

12. **Fourier transform of integrals:** Show that if $f(x)$ has a Fourier transform $\tilde{f}(k)$, then the Fourier transform of its integral

$$g(x) = \int_{-\infty}^{x} \mathrm{d}y \; f(y)$$

is given by

$$\tilde{g}(k) = -\frac{\mathrm{i}}{k}\tilde{f}(k) + \pi \tilde{f}(0)\delta(k)$$

13. **Symmetry properties of DFT:** The functions $f(t_k)$ and $\tilde{f}(\omega_l)$ are discrete Fourier transforms of each other:

$$\tilde{f}(\omega_l) = \frac{T}{N}\sum_{k=0}^{N-1} f(t_k)\; \mathrm{e}^{-\mathrm{i}\omega_l t_k}, \qquad f(t_k) = \frac{1}{T}\sum_{l=0}^{N-1} \tilde{f}(\omega_l)\; \mathrm{e}^{\mathrm{i}\omega_l t_k}$$

Derive the following symmetry relations:

(i) If $f(t_k)$ is real, then

$$\tilde{f}(\omega_l) = \tilde{f}^* \left(\frac{2\pi N}{T} - \omega_l\right)$$

(ii) If $f(t_k)$ is pure imaginary, then

$$\tilde{f}(\omega_l) = -\tilde{f}^* \left(\frac{2\pi N}{T} - \omega_l\right)$$