# Mathematical Physics I: Mid-term Examination

**HRI Graduate School**

**August - December 2010**

**07 October 2010**

**Duration: 3 hours**

- The paper is of 100 marks. Attempt all the questions.
- You are free to consult your class notes during the examination.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. Compute the product in the given ring.
(i) [12][16] in $\mathbb{Z}_{24}$
(ii) [16][3] in $\mathbb{Z}_{32}$
(iii) [11][-4] in $\mathbb{Z}_{15}$
(iv) [20][-8] in $\mathbb{Z}_{26}$
(v) $([2],[3])\,([3],[5])$ in $\mathbb{Z}_5 \times \mathbb{Z}_9$
(vi) $([-3],[5])\,([2],[-4])$ in $\mathbb{Z}_4 \times \mathbb{Z}_{11}$

$[2.5 \times 4 + 3.5 \times 2 = 17]$

2. Prove the following results for a group $(G, *)$.
(i) The identity element $e$ is unique.
(ii) Each $a \in G$ has a unique inverse $a^{-1}$.

$[4 + 4 = 8]$

3. Show that every element of $SU(2)$ has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a = d^*$ and $b = -c^*$. Remember that $SU(2)$ is the group of $2 \times 2$ unitary matrices with determinant 1 and $^*$ denotes complex conjugation.

[6]

4. The **commutator** $[f, g]$ of two operators in $L(V, V)$ is defined as $[f, g] = fg - gh$. Prove the **Jacobi identity** $[[f, g], h] + [[g, h], f] + [[h, f], g] = 0$.

[6]

5. In the following, determine whether the vector spaces $V$ and $W$ are isomorphic. Justify your answers.
(i) Let $V = \{\mathsf{A} \in M^{3,3}(\mathbb{R}) \mid \mathsf{A} = \mathsf{A}^T\}$ and $W = \{\mathsf{A} \in M^{3,3}(\mathbb{R}) \mid \mathsf{A} = -\mathsf{A}^T\}$. $\mathsf{A}^T$ denotes the transpose of $\mathsf{A}$.
(ii) Let $V = \{f(t) \in P_5(\mathbb{R}) \mid f(t) = f(-t)\}$ and $W = P_3(\mathbb{R})$. Note that $P_n(\mathbb{R})$ is the space of polynomials of degree $\leq n$ having real coefficients.
(iii) Let $V = L\left(P_2(\mathbb{R}), M^{2,2}(\mathbb{R})\right)$ and $W = L(M^{2,3}(\mathbb{R}), \mathbb{R}^2)$.

$[2 + 2 + 2 = 6]$

6. Let $f$ be a linear operator on $V$. One can define the exponential of a linear operator through the convergent infinite series

$$\mathrm{e}^f \equiv \exp(f) = \sum_{j=0}^{\infty} \frac{f^j}{j!}$$

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x, y) = (-y, x)$. Show that

$$\mathrm{e}^{\alpha f} = f \sin\alpha + 1 \cos\alpha$$

where $\alpha$ is a scalar.

What is the result of applying $\mathrm{e}^{\alpha f}$ on $(x, y)$?

[8]

7. Let $\mathbf{v} \in \mathbb{R}^3$ be a nonzero vector. Show that $\Pi_v : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\Pi_v(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$ is a homomorphism of vector spaces over $\mathbb{R}$, where $\mathbf{v} \times \mathbf{w}$ is the vector (cross) product of $\mathbf{v}$ and $\mathbf{w}$. Determine the kernel and the image of $\Pi_v$.

[6]

8. Let $E$ be a linear operator on $V$ such that $E^2 = E$. Such an operator is termed a projection. Let $U$ be the image of $V$ and $W$ be the kernel.

(i) If $\alpha \in U$, then show that $E(\alpha) = \alpha$, i.e., $E$ is the identity map on $U$.

(ii) If $E \neq 1$, then show that $E(\beta) = 0$ for some $\beta \neq 0 \in V$, i.e., the kernel is not simply the set $\{0\}$.

(iii) Show that $U \cap W = \{0\}$.

$[3 + 3 + 2 = 8]$

9. Let $V = \mathbb{C}^4$ with the standard inner product. Let $f$ be the linear operator on $V$ whose matrix with respect to the standard basis for $V$ is given by:

$$\mathsf{f} = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

(i) Find an orthonormal basis for $V$ consisting of eigenvectors of $f$.
(ii) Find an orthogonal matrix $\mathsf{P}$ and a diagonal matrix $\mathsf{D}$ such that $\mathsf{D} = \mathsf{P}^{-1}\mathsf{f}\mathsf{P}$.

$[4 + 4 = 8]$

10. Let $V$ be the vector space of $2 \times 2$ matrices with the usual basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right\}$$

Let $\mathsf{M} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $g$ be the linear operator on $V$ defined by $g(\mathsf{A}) = \mathsf{MA}$. Find the matrix representation of $g$ relative to the above usual basis of $V$.

[6]

11. Let $V = \mathbb{C}^4$ with the standard inner product. Let

$$W = \{x = (x_1, x_2, x_3, x_4) \in V | \sqrt{2}x_1 - x_3 = 0, x_1 - \mathrm{i}x_2 + x_4 = 0\}.$$

(i) Find an orthonormal basis for $W$.
(ii) Find an orthonormal basis for $W^{\perp}$.

$[7 + 7 = 14]$

12. Let $V$ be the vector space of polynomials $\alpha(t)$ having degree $\leq 2$ with the inner product $\langle \alpha, \beta \rangle = \int_1^1 \mathrm{d}t \; \alpha(t)\beta(t)$. Start with the usual basis $\{1, t, t^2\}$ and apply Gram-Schmidt algorithm to obtain an orthogonal basis $\{f_0, f_1, f_2\}$. Normalise the new basis such that $f_i(t = 1) = 1$. Can you identify the series of polynomials?

[7]